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EDUCATION

MATHEMATICS

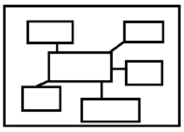
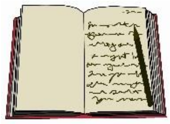
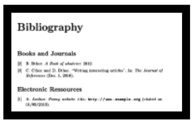
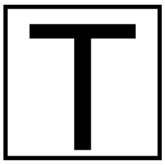
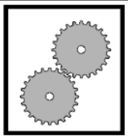
WINTER CLASSES

GRADE 12

TERM 2

TEACHER AND LEARNER CONTENT MANUAL

ICON DESCRIPTION

 MIND MAP	 EXAMINATION GUIDELINE	 CONTENTS	 ACTIVITIES
 BIBLIOGRAPHY	 TERMINOLOGY	 WORKED EXAMPLES	 STEPS

CONTENTS **PAGE**

FUNCTIONS AND GRAPHS

Overview:

Revise the effect of the parameters a and q and investigate the effect of p on the graphs of the functions defined by: 1.1. $y = f(x) = a(x+p)^2 + q$ 1.2. $y = f(x) = \dfrac{a}{x+p} + q$ 1.3. $y = f(x) = ab^{x+p} + q$ where $b > 0, b \neq 1$	Investigate numerically the average gradient between two points on a curve and develop an intuitive understanding of the concept of the gradient of a curve at a point.	1. Definition of a *function.* 2. General concept of the *inverse of a function* and how the domain of the function may need to be restricted (in order to obtain a one-to-one function) to ensure that the inverse is a function. 3. Determine and sketch graphs of the inverses of the functions defined by $y = ax + q$; $y = ax^2$ $y = b^x$; $(b > 0, b \neq 1)$ Focus on the following characteristics: domain and range, intercepts with the axes, turning points, minima, maxima, asymptotes (horizontal and vertical), shape

(SOURCE: CURRICULUM AND ASSESSMENT POLICY STATEMENT(CAPS)SENIOR PHASE GRADES (10 – 12) MATHEMATICS)

Important terminilogy

Domain:	the set of possible x-values	**For all functions**
Range:	the set of possible y-values	
Axis of symmetry:	an imaginary line that divides a graph into two mirror images of each other.	**See the hyperbola and parabola**
Maximum:	the highest possible y-value of a function.	**See the parabola**
Minimum:	the lowest possible y-value of a function.	
Asymptote:	an imaginary line that a graph approaches but never touches.	**See the hyperbola and exponential function**
Turning point:	The point at which a graph reaches its maximum or minimum value and changes direction.	**See the parabola**

The concepting of increasing and decresing in functions: all functions

- The function is INCREASING when the value of y increases as x is increasing from left to right
  - **THE GRAPH GOES UP**

- The function is DECREASING when the value of y decreases as x is increasing from left to right
  - **THE GRAPH GOES DOWN**

SECTION 1: LINEAR FUNCTION (STRAIGHT LINE)

The graph of $y = mx + c$

Standard form of linear function

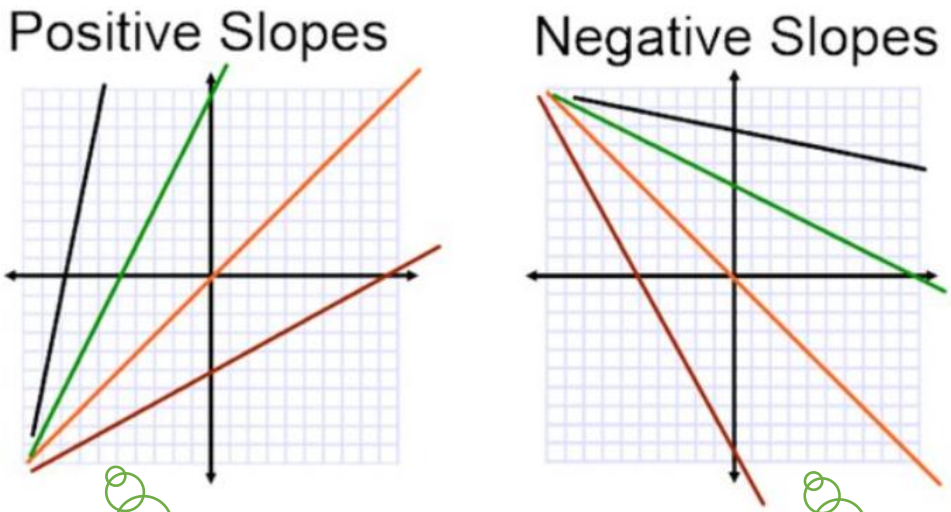
WHERE

$m = gradient$

When $m > 0$ ***(gradient is positive)*** and
$m < 0$ ***(gradient is negative)***

Domain: $x \in R$
Range: $y \in R$

Shape



When $m > 0$

1. **The gradient is positive**
2. **The function is increasing**

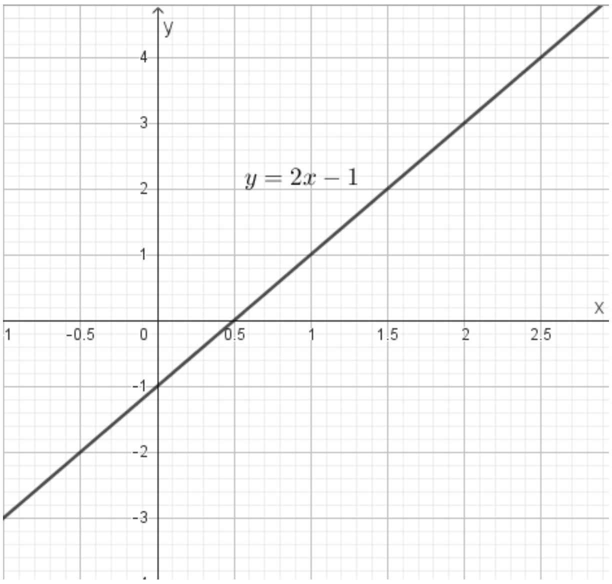
When $m < 0$

3. **The gradient is negative**
4. **The function is**

Example 1

Sketch the graph of $y = 2x - 1$ and determine the domain and range of the function, and state if the function is increasing or decreasing.

Solution



$\boldsymbol{x - intercept: ley\ y = o}$

$0 = 2x - 1$

$-2x = -1$

$x = \frac{1}{2}$

$\therefore \left(\frac{1}{2}; 0\right)$

$y - intercept: let\ x = 0$

$y = 2(0) - 1$

$y = -1$

$\therefore (0; -1)$

Domain: $x \in R$

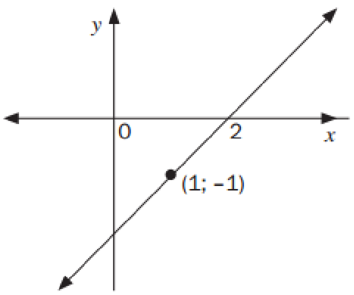
Range:$y \in R$

The function is increasing

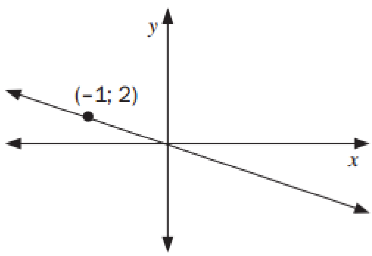
Example 2

Determine the equations of the following graphs:

1.



2.



Solutions

1.

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-1 - 0}{1 - 2}$$
$$a = 1$$
$$\therefore y = 1x + c$$
$$0 = 1(2) + c$$
$$c = -2$$

$$y = x - 2$$

2.

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2 - 0}{-1 - 0}$$
$$a = -2$$
$$\therefore y = -2x + c$$
$$0 = -2(0) + c$$
$$c = 0$$

$$y = -2x$$

SECTION 2: HYPERBOLIC FUNCTIONS(HYPERBOLA)

The graph of $y = \frac{a}{x+p} + q$

Standard form of hyperbola

$$\textit{take note that } y = \frac{2}{x-2} + 1$$

$$= \frac{2}{x+(-2)} + 1$$

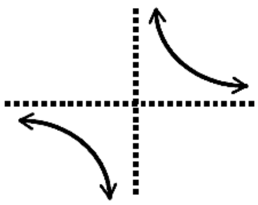
The equations of asymptotes are $x = -p$ ***(vertical asymptote)*** and $y = q$ ***(horizontal asymptote)***

Domain: $x \in R, x \neq -p$

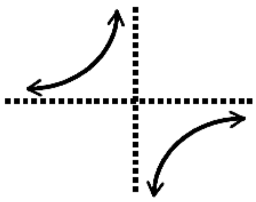
Range: $y \in R, y \neq q$

Shape

If $a > 0$ then the graph decreases for all $x < 0$ or $x > 0$.



If $a < 0$ then the graph increases for all $x < 0$ or $x > 0$.



The equations of the axis of symmetry

The hyperbola has two equations of symmetry

$m = 1$	$m = -1$
$y = x + c$	$y = -x + c$

N.B the equations of the axis of symmetry of the hyperbola passes through the point of intersection of asymptotes $(-p; q)$

In general, for the hyperbola, the equations of the axis of symmetry are given by the following formulae:

$m = 1$	$m = -1$
$y = (x + p) + q$ $\therefore y = x + p + q$	$y = -(x + p) + q$ $\therefore y = -x - p + q$

N.B Ensure that the hyperbola is in standard form before applying the formula

Example 1

Sketch the graph of $y = \frac{10}{x+2} - 3$, write down the domain and range of the function, and write down the equations of asymptotes. state whether the function is increasing or decreasing. Lastly, determine the equation of the asymptote with a positive gradient.

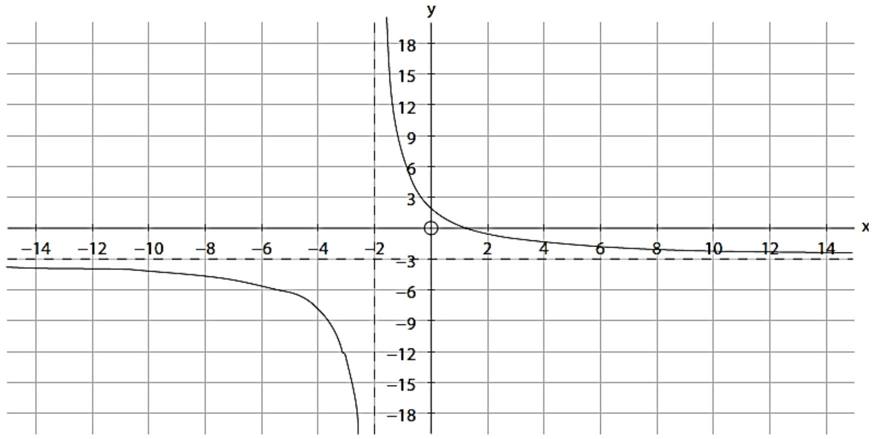
Solutions

The asymptotes are $x = -2$ and $y - 3$. (These are read from the equation).

x-intercept: Let $y = 0$:

$0 = \frac{10}{x+2} - 3 \therefore 10 = 3(x+2) \quad \therefore 3x = 4 \quad \therefore x = \frac{4}{3}$

y-intercept: Let $x = 0$: $y = \frac{10}{0+2} - 3 = 5 - 3 = 2$



graph not drawn to scale

Domain: $x \in R, x \neq -2$

Range: $y \in R, y \neq -3$

The function is decreasing

<u>Equation of the axis of symmetry with positive gradient:</u>

$y = x + c$

$-3 = -2 + c$

$c = -1$

$\therefore y = x - 1$

Example 2

Sketch the graph of $y = -1 - \frac{8}{x-4}$, and write down the domain and range of the function. And determine the equation of the axis of symmetry with a negative gradient.

Solution

This equation can also be written as $y = \frac{-8}{x-4} - 1$.

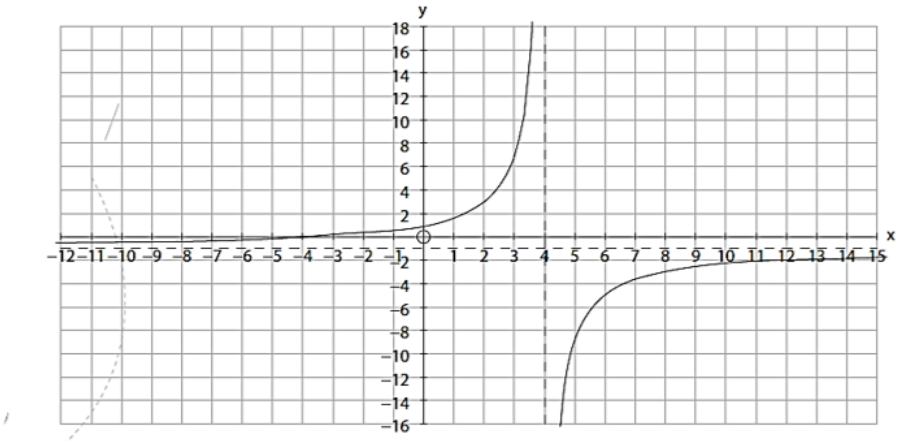
Standard form of hyperbola

Asymptotes: $x = 4$ and $y = -1$

y-intercept: Let $x = 0$: $y = -1 - \frac{8}{0-4} = -1 + 2 = 1$

x-intercept: Let $y = 0$: $0 = -1 - \frac{8}{x-4}$ $\quad \therefore (x-4) = -8 \quad \therefore x = -4$

The hyperbola will look as follows:



graph not drawn to scale

Domain: $x \in R, x \neq 4$
Range: $y \in R, y \neq -1$

The function is increasing

<u>Equation of the axis of symmetry with positive gradient:</u>

$y = -x + c$

$-1 = -(4) + c$

$c = 3$

$\therefore y = -x + 3$

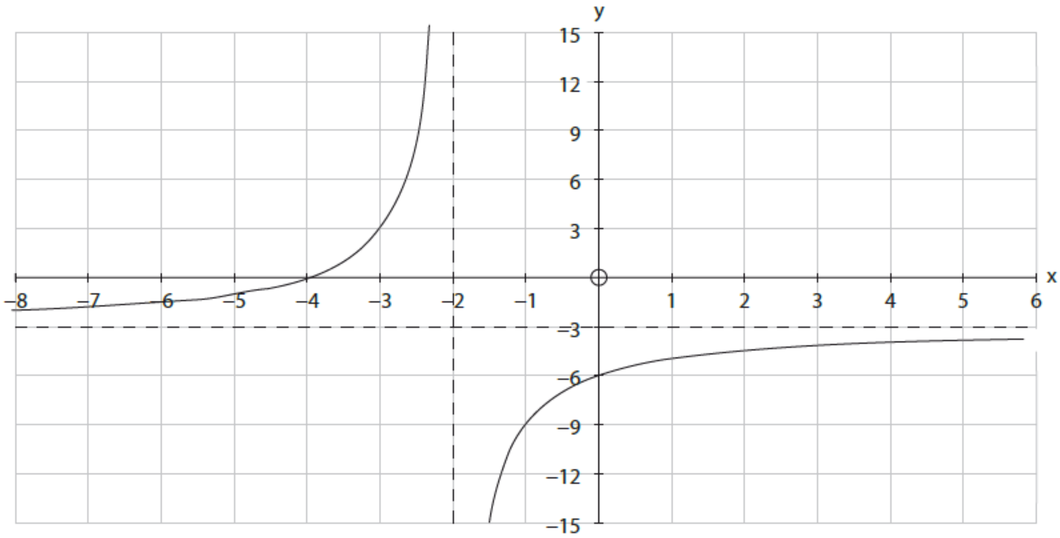
Example 3

Determining the equation of a hyperbola when the graph is given

Step 1: The position of the asymptotes give us the value(s) of p and q in $y = \frac{a}{x+p} + q$.

Step 2: To find the value of a, we substitute any point on the graph into the equation.

EXAMPLE: The graph of $y = \frac{a}{x+p} + q$ is sketched below. Determine the value(s) of a, p and q.



From the graph we see that, $p = 2$ and $q = -3$.

The equation becomes: $y = \frac{a}{x+2} - 3$.

Substituting (0; −6) or any other point on the graph into the equation gives:

$-6 = \frac{a}{0+2} - 3 \quad \therefore -12 = a - 6 \quad \therefore a = -6$

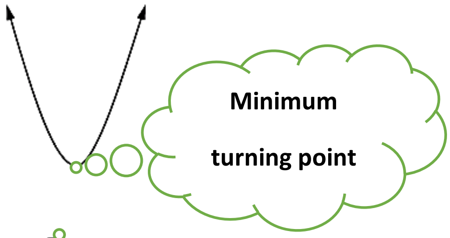
$$\therefore \boldsymbol{y = -\frac{6}{x+2} - 3}$$

$\boldsymbol{N.B\ the\ function\ is\ increasing}$

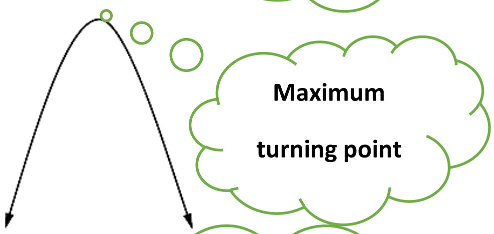
SECTION 3: QUADRATIC FUNCTION(PARABOLA)

The graph of $y = a(x+p)^2 + q$

If a is positive, i.e. a > 0, then the shape of the graph is ☺.



If a is negative, i.e. a < 0, then the shape of the graph is ☹.



The graph has the axis of symmetry at $x = -p$

The graph has the turning point by $(-p\ ;\ q)$

Domain: $x \in R$

Range: $y \geq q$) 😊 **(WHEN a > 0)**
or
$y \leq q$ 😟 **(WHEN a < 0)**

N.B

1. q is the minimum of the parabola when $a > 0$
2. q is the maximum of the parabola when $a < 0$

N.B The parabola changes from increasing to decreasing or decreasing to increasing at the turning point.

when a > 0
$\mathbf{1.}\ \boldsymbol{The\ graph\ increases\ for: x > -p}$ $\mathbf{2.}\ \boldsymbol{The\ graph\ decrease\ for: x < -p}$
when a < 0
$\mathbf{1.}\ \boldsymbol{The\ graph\ increases\ for: x < -p}$ $\mathbf{2.}\ \boldsymbol{The\ graph\ decrease\ for: x > -p}$

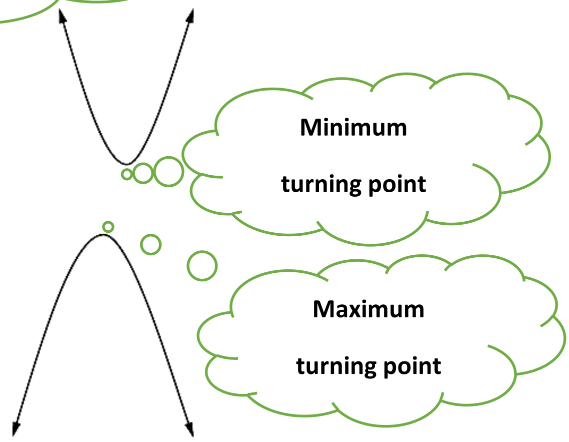
The quadratic function can also be represented in the form:

$y = f(x) = ax^2 + bx + c$

Standard form of parabola

If a is positive, i.e. a > 0, then the shape of the graph is ☺.

If a is negative, i.e. a < 0, then the shape of the graph is ☹.



The graph has the axis of symmetry at $x = -\frac{b}{2a}$

The graph has the turning point by $(-\frac{b}{2a}\ ;\ f(-\frac{b}{2a}))$

Domain: $x \in R$

Range: $y \geq f(-\frac{b}{2a})$ 😊 **(WHEN a > 0)**
or
$y \leq f(-\frac{b}{2a})$ 😞 **(WHEN a < 0)**

N.B

1. $f\left(-\frac{b}{2a}\right)$ **is the minimum of the parabola when** $a > 0$

2. $f\left(-\frac{b}{2a}\right)$ **is the maximum of the parabola when** $a < 0$

Example 1

Sketch the graph of $f(x) = x^2 - 5x - 6$, clearly showing intercepts with the axes and the turning point. Also give the interval where the function is increasing and where the function is decreasing. Lastly, determine the domain and range of of $f(x) = x^2 - 5x - 6$.

Solutions

Sketch the graph of $f(x) = x^2 - 5x - 6$

1. **y-intercept**

   $f(0) = -6$

   Therefore the co-ordinates of the y-intercept are $(0; -6)$

2. **x-intercept**

   $x2 - 5x - 6 = 0$

   $(x - 6)(x + 1) = 0$

   $x = 6$ or $x = -1$

   $(6; 0)$ and $(-1; 0)$

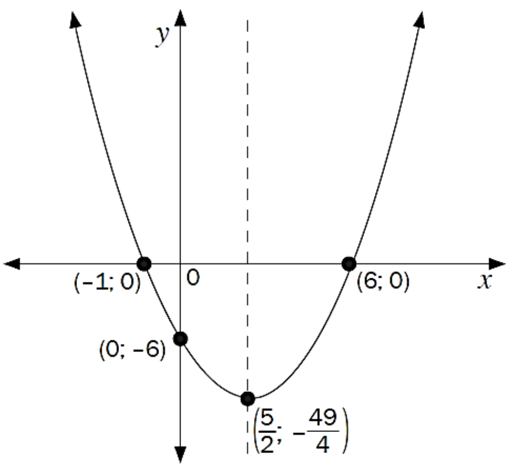
3. **Axis of symmetry**

$$x = \frac{-b}{2a}$$
$$= \frac{-(-5)}{2(1)}$$
$$= \frac{5}{2}$$

4. **Turning point**

$$f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) - 6$$
$$= -12\frac{1}{4}$$
$$\therefore TP\left(\frac{5}{2}; -12\frac{1}{4}\right)$$

5. **Sketch graph**



6. The interval where the graph is increasing

$x > \frac{5}{2}$

The interval where the graph is decreasing

$x < \frac{5}{2}$

7. Domain: $x \in R$

Range: $y \geq -\frac{49}{4}$

Determining the equation of the quadratic function

Given the x-intercepts and one point	Given the turning point and one point
• Use the formula: $y = a(x - x_1)(x - x_2)$. • Substitute the values of the x-intercepts. • Substitute the given point which is not the x-intercept. • Solve for a. • Write the equation in the form $f(x) = ax^2 + bx + c$.	• Use the formula: $y = a(x + p)^2 + q$. • Substitute the co-ordinates of the turning point $(p; q)$. • Substitute the given point. • Solve for a. • Write the equation in the form $y = a(x + p)^2 + q$ or $f(x) = ax^2 + bx + c$ depending on the instruction in the question.

Given the co-ordinates of three points on the parabola
• Use the formula: $y = ax^2 + bx + c$. • One of the given point is the y-intercept, therefore c is given, so substitute its value. • Substitute the co-ordinates of the other two points into $y = ax^2 + bx + c$. • Solve the two equations simultaneously for a and b.

SOURCE:Mind The Gap Mathematics Study Guide

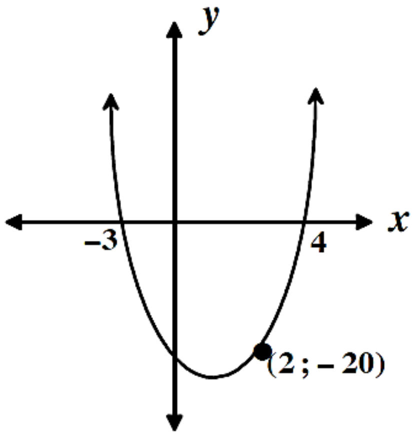
Example 2

Determine the equation of the parabola in the form $f(x) = ax^2 + bx + c$.

$y = a(x - x_1)(x - x_2)$

$\therefore y = a(x - (-3))(x - 4)$

$\therefore y = a(x + 3)(x - 4)$



Substitute $(2\,;-20)$ to find the value of a.

$-20 = a(2+3)(2-4)$

$\therefore -20 = a(5)(-2)$

$\therefore -20 = -10a$

$\therefore a = 2$

$\therefore y = 2(x+3)(x-4)$

$\therefore y = 2(x^2 - x - 12)$

$\therefore f(x) = 2x^2 - 2x - 24$

Example 3

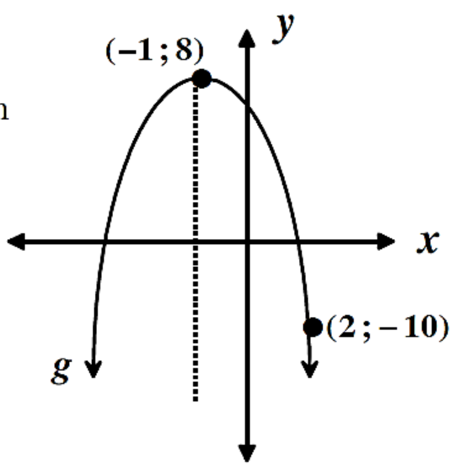
Determine the equation of g in the form $y = ax^2 + bx + c$

The axis of symmetry is given by $x = -1$.

$\therefore x + 1 = 0$

From the work on parabolas, it is clear that the expression $x + 1$ is in the brackets of the equation for a parabola.

Also, the value of q is 8 (the y-value of the turning point).



$\therefore y = a(x+1)^2 + 8$

Now substitute the point $(2\,;-10)$ which lies on the graph of the parabola.

$-10 = a(2+1)^2 + 8$

$\therefore -18 = a(3)^2$

$\therefore -18 = 9a$

$\therefore a = -2$

$\therefore y = -2(x+1)^2 + 8$

$\therefore y = -2(x^2 + 2x + 1) + 8$

$\therefore y = -2x^2 - 4x - 2 + 8$

$\therefore y = -2x^2 - 4x + 6$

SECTION 4: EXPONENTIAL FUNCTION

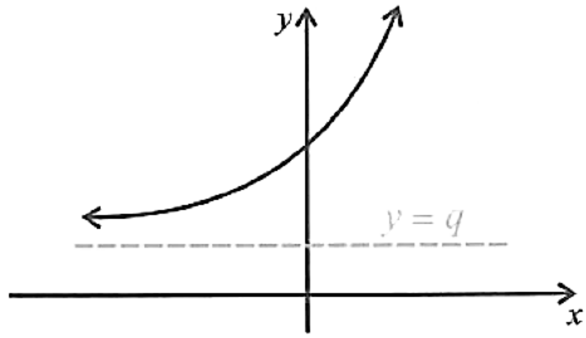
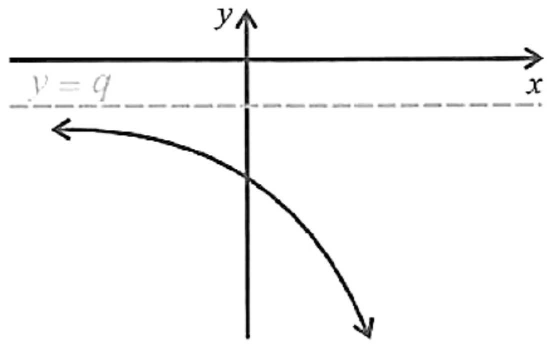
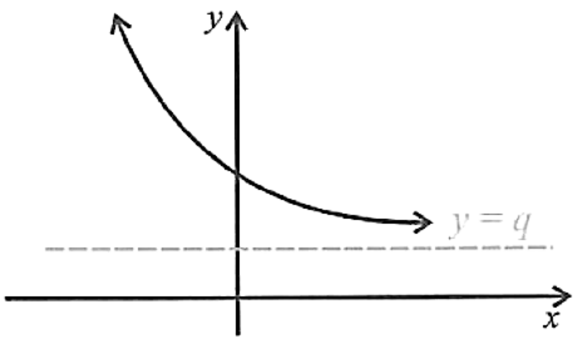
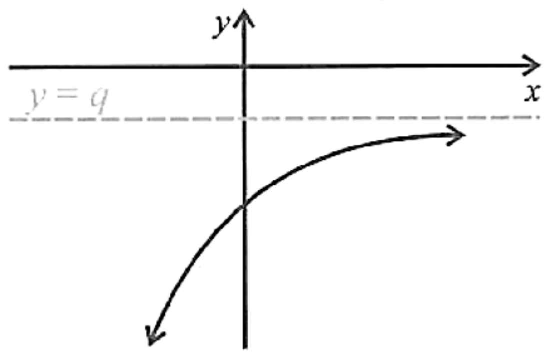
The graph of $y = a.b^{x+p} + q$ where $b > 0$ and $b \neq 1$

Standard form of exponential function

The equation of an asymptote is $y = q$ ***(horizontal asymptote)***

Domain: $x \in R$

Range: $y > q$ [if $a > 0$] or $y < q$ [if $a < 0$]

$a > 0$ and $b > 1$	$a < 0$ and $b > 1$
The graph lies above the horizontal asymptote and is an increasing function.	The graph lies below the horizontal asymptote and is a decreasing function.
	
$a > 0$ and $0 < b < 1$	**$a < 0$ and $0 < b < 1$**
The graph lies above the horizontal asymptote and is a decreasing function.	The graph lies below the horizontal asymptote and is an increasing function.
	

(Source: MATHS MADE EASY – A comprehensive guide to Grade 12 Mathematics)

Example 1

Sketch the graph of $y = 2\left(\frac{1}{2}\right)^{x+1} - 4$

Horizontal asymptote: $y = -4$

x-intercept: let $y = 0$:

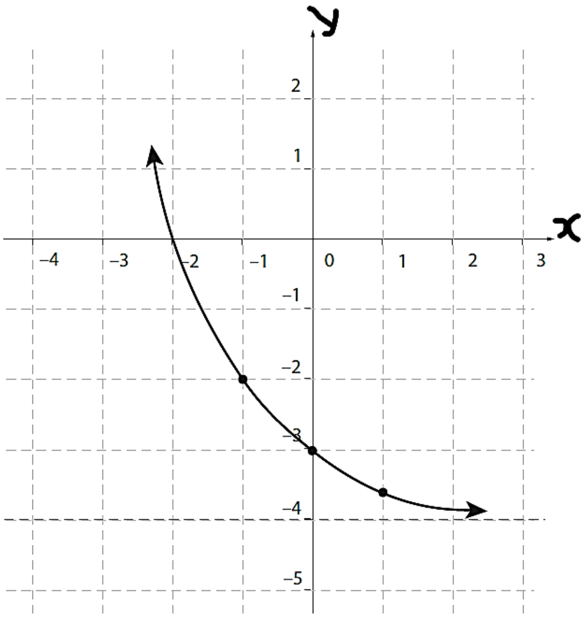
$0 = 2\left(\frac{1}{2}\right)^{x+1} - 4 \therefore \left(\frac{1}{2}\right)^{x+1} = 2$

$\left(\frac{1}{2}\right)^{x+1} = \left(\frac{1}{2}\right)^{-1}$

$\therefore x + 1 = -1 \quad \therefore x = -2$

y-intercept: let x = 0

y = - 3



N.B the function is decreasing

Example 2

Let us look at a second example, $y = 3{\cdot}(3)^{x-2} + 1$

Horizontal asymptote: $y = 1$

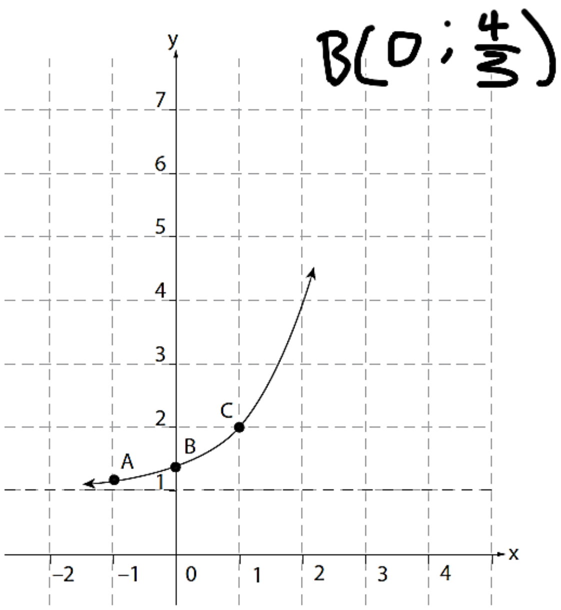
x-intercept: $0 = 3{\cdot}(3)^{x-2} + 1$

$\therefore 3(3)^{x-2} = -1$

$\therefore$ No x-intercept

y-intercept: let x = 0

y = $\frac{4}{3}$

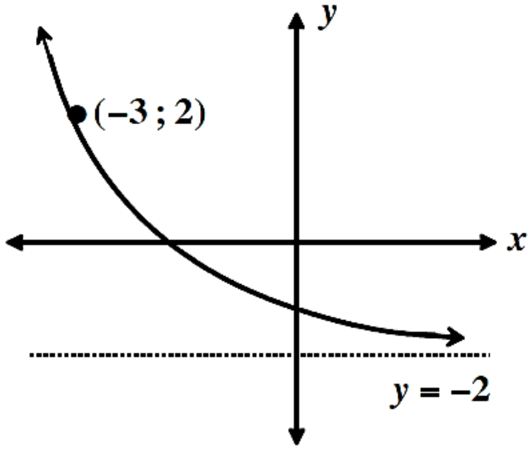


N.B the function is inccreasing

Determining the equation of exponential function

Example 3

Determine the equation of $g(x) = b^{x+1} + q$



Horizontal asymptote: $y = -2$

Therefore the value of q is -2

$\therefore y = b^{x+1} - 2$

Now substitute the point $(-3\,;2)$ into the equation to get b.

$2 = b^{-3+1} - 2$

$\therefore 4 = b^{-2}$

$\therefore 4 = \dfrac{1}{b^2}$

$\therefore 4b^2 = 1$

$\therefore 4b^2 - 1 = 0$

$\therefore (2b+1)(2b-1) = 0$

$\therefore b = -\dfrac{1}{2}$ or $b = \dfrac{1}{2}$

But $b \neq -\dfrac{1}{2}$

$\therefore b = \dfrac{1}{2}$

Note:
Since $b > 0$, you could have solved the equation $4b^2 = 1$ as follows:

$4b^2 = 1$

$\therefore b^2 = \dfrac{1}{4}$

$\therefore b = \dfrac{1}{2}$

Therefore the equation is: $g(x) = \left(\dfrac{1}{2}\right)^{x+1} - 2$

<u>**TRANSFORMATION OF FUNCTIONS**</u>

REFLECTIONS AND TRANSLATIONS

Given $f(x) = \frac{2}{x+1} - 3$	**Given $f(x) = 2.3^{x-2} + 4$**	**Given $f(x) = x^2 + 5x + 6$**
a. The graph of $g(x)$ is obtained by shifting the graph of $f(x)$ 2 units up and 3 units to left. Determine the equation of $g(x)$. **Solution** $g(x) = f(x+3) + 2$ $= \frac{2}{x+3+1} - 3 + 2$ $= \frac{2}{x+4} - 1$ b. The graph of $h(x)$ is obtained by reflecting the graph of $f(x)$ in the $x - axis$. Determine the equation of $h(x)$. **Solution** $h(x) = -f(x)$ $= -\left(\frac{2}{x+1} - 3\right)$ $= -\frac{2}{x+1} + 3$ c. The graph of $m(x)$ is obtained by reflecting the graph of $f(x)$ in the $y - axis$. Determine the equation of $m(x)$. **Solution** $m(x) = f(-x)$ $= \frac{2}{-x+1} - 3$ $= \frac{2}{-(x-1)} - 3$ $= -\frac{2}{x-1} - 3$	a. The graph of $g(x)$ is obtained by shifting the graph of $f(x)$ 2 units up and 3 units to left. Determine the equation of $g(x)$. **Solution** $g(x) = f(x+3) + 2$ $= 2.3^{x+3-2} + 4 + 2$ $= 2.3^{x+1} + 6$ b. The graph of $h(x)$ is obtained by reflecting the graph of $f(x)$ in the $x - axis$. Determine the equation of $h(x)$. **Solution** $h(x) = -f(x)$ $= -(2.3^{x-2} + 4)$ $= -2.3^{x-2} - 4$ c. The graph of $m(x)$ is obtained by reflecting the graph of $f(x)$ in the $y - axis$. Determine the equation of $m(x)$. **Solution** $m(x) = f(-x)$ $= 2.3^{-x-2} + 4$ $= 2.3^{-(x+2)} + 4$ $= 2.\left(\frac{1}{3}\right)^{x+2} + 4$	a. The graph of $g(x)$ is obtained by shifting the graph of $f(x)$ 2 units down and 3 units to right. Determine the equation of $g(x)$. **Solution** $g(x) = f(x-3) - 2$ $= (x-3)^2 + 5(x-3) + 6 - 2$ $= x^2 - 6x + 9 + 5x - 15 + 4$ $= x^2 - x - 2$ b. The graph of $h(x)$ is obtained by reflecting the graph of $f(x)$ in the $x - axis$. Determine the equation of $h(x)$. **Solution** $h(x) = -f(x)$ $= -(x^2 + 5x + 6)$ $= -x^2 - 5x - 6$ c. The graph of $m(x)$ is obtained by reflecting the graph of $f(x)$ in the $y - axis$. Determine the equation of $m(x)$. **Solution** $m(x) = f(-x)$ $= (-x)^2 + 5(-x) + 6$ $= x^2 - 5x + 6$

SECTION 5: INVERSE FUNCTIONS

The concept of a function

A function f, is defined as a relationship between values, where each input value maps to one output value.
In other words, for an equation to be called a function, there can only be one y - value for a particular x-value.

There are two types of functions:
1. **One-to-One Functions**
2. **Many-to-One Functions**

ONE-TO-ONE FUNCTIONS

A one-to-one **function is a function where there is a single y-value for a particular x-value.**

MANY-TO-ONE FUNCTIONS

A function cannot have more than one y value to each x value. However, a function can have more than one x value for a particular y value. **These are known as** many to-one **functions.**

VERTICAL LINE TEST

To test if a graph is a function, use the vertical line **test**. If a vertical line (a line parallel to the y-axis) touches the graph more than once at any point, the graph is not a function. You don't have to draw a line, just hold a ruler parallel to the y-axis and move it along the graph. If the ruler touches the graph more than once for a single x value, anywhere on the graph, then the graph is not a function. In the case of the graph not being a function, it is said to be a relation.

HORIZONTAL LINE TEST

If a graph passes the vertical line test, it is a function. **The** horizontal line **test can be used to determine what type of function the graph represents.**

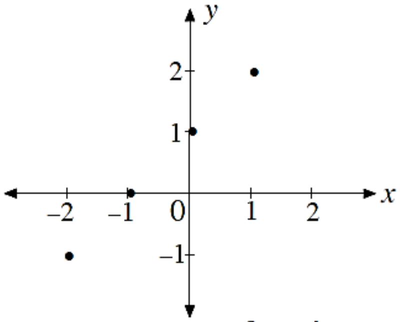
If a horizontal line (a line parallel to the x-axis) is drawn and moved along the graph and it **touches the graph more than once** at any point, it is a **many-to-one function** (many x-values to a single y-value). Otherwise, it is a one-to-one function.

(Source: MATHS MADE EASY – A comprehensive guide to Grade 12 Mathematics)

Examples

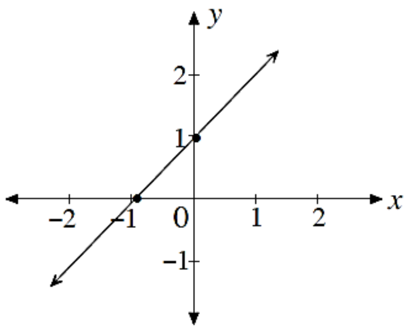
Determine whether the following relations are functions or not. If the graph is a function, determine whether the function is one-to-one or many-to-one.

(a)



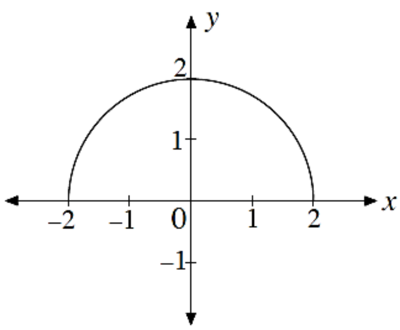
one-to-one function

(b)



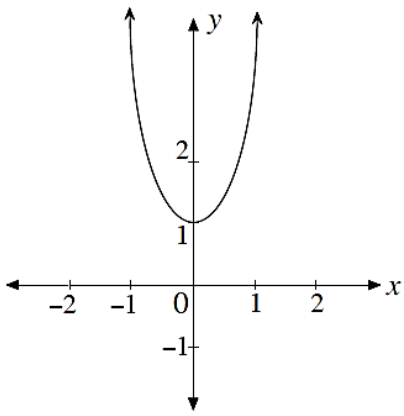
one-to-one function

(c)



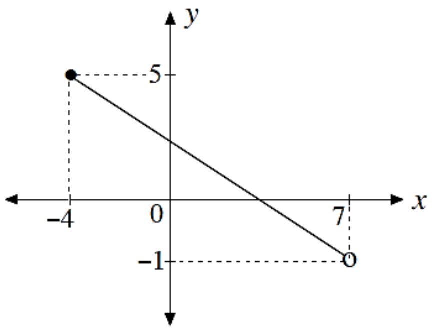
many-to-one function

(d)



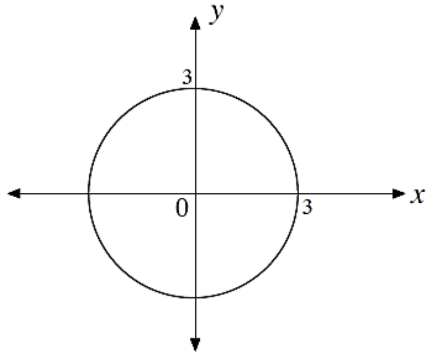
many-to-one function

(e)



one-to-one function

(f)



not a function

(Source: Mind Action series Mathematics12 textbook and Workbook **)**

Inverse Function

- The inverse of a function takes the y-values (range) of the function to the corresponding x-values (domain) and vice versa. Therefore the x and y values are interchanged.
- The function is reflected along the line $y = x$ to form the inverse.
- The notation for the inverse of a function is f^{-1}.

- N.B The domain of the inverse is the range of the function and the range of the inverse is the domain of the function.
- When the function is increasing, its inverse also increases. When the function decreases, its inverse will also decrease.

Inverse function : Linear $(y = ax + q)$

Example 1

Given $f(x) = 2x + 6$.

1. Determine $f^{-1}(x)$
2. Sketch the graphs of $f(x)$, $f^{-1}(x)$ and $y = x$ on the same set of axis

Solutions

1. In order to find the inverse of a function, there are two steps:

STEP 1: Swap the x and y

$$y = 2x + 6$$

becomes $x = 2y + 6$

We then rewrite the equation to make y the subject of the formula. Therefore,

STEP 2: make y the subject of the formula

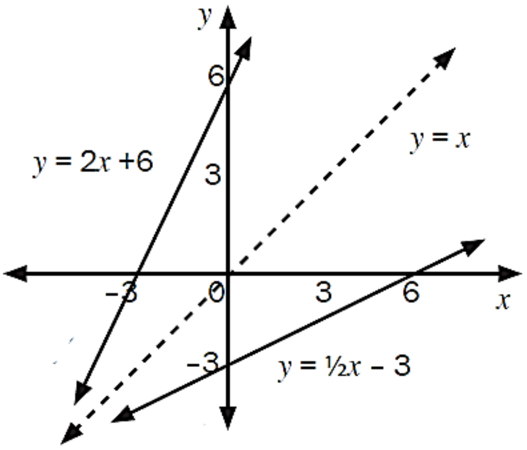
$x = 2y + 6$

$x - 6 = 2y$

So $y = \frac{1}{2}x - 3$

We can say that the inverse function $f^{-1}(x) = \frac{1}{2}x - 3$

2.



Inverse function : Quadratic ($y = ax^2$)

Example 2

a) Sketch $f(x) = 2x^2$

b) Determine the inverse of $f(x)$

c) Sketch $f^{-1}(x)$ and $y = x$ on the same axes as $f(x)$

a) & c)

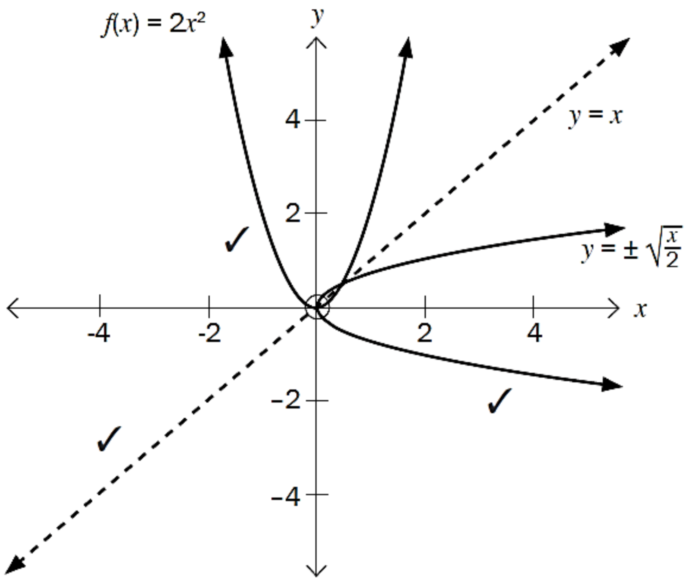
Solution

1. b) $y = 2x^2$

$x = 2y^2$ ✓

$y = \pm\sqrt{\frac{x}{2}}$ ✓

- This is not a function.
- Check it with a vertical line test. There are two y-values for one x-value.
- Not all inverses of functions are also functions. Some inverses of functions are relations.
- If an inverse is not a function, then we can restrict the **domain** of the **function** in order for the inverse to be a function.



- To make the inverse a function, we need to choose a set of x-values in the function and work only with those. We call this **'restricting the domain'**.
- A one to one function has an inverse that is a function

  Example: $y = 3x + 4$ is a one to one function. For every x value there is one and only one y value

- A many to one function has an inverse that is not a function. However, we can restrict the domain of the function to make its inverse a function.

  Example: $y = 2x^2$ is a many to one function. For two or many x values there is one y value. (if $x = 2$, then $y = 8$.
  If $x = -2$, then $y = 8$). Therefore, its inversey $= \pm\sqrt{\frac{x}{2}}$, is not a function.
- To check for a function, draw a vertical line. If any vertical line cuts the graph in only one place, the graph is a function.
  If any vertical line cuts the graph in more than one place, then the graph is not a function.
- To check for a one-to-one function, draw a horizontal line. If any horizontal line cuts the graph in only one place, the graph is a one-to-one function. If any horizontal line cuts the graph in more than one place, then the graph is a many-to-one function.

Example 3

Given: $g(x) = -x^2$ where $x \leq 0$

(a) Write down the inverse of g, g^{-1} in the form $h(x) = ...$

(b) Sketch the graphs of g, h and $y = x$ on the same set of axis.

Solutions

(a) $y = -x^2$

$x = -y^2$

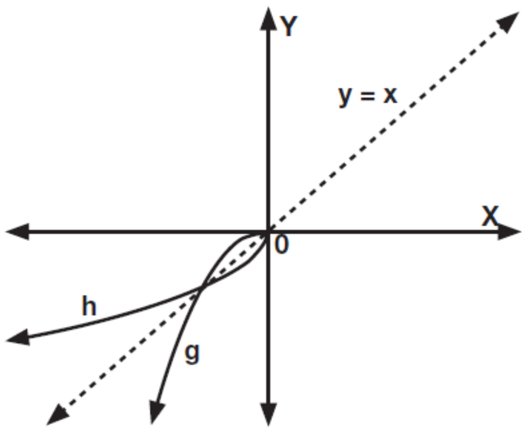
$-x = y^2$

$\pm\sqrt{-x} = y$

$-\sqrt{-x} = y$ where $x \leq 0$

$\therefore h(x) = -\sqrt{-x}$

(b)



Inverse function : Exponential $y = b^x; (b > 0, b \neq 1)$

Example 4

Given: $f(x) = 2^x$

a) Determine f^{-1} in the form $y = \ldots\ldots$

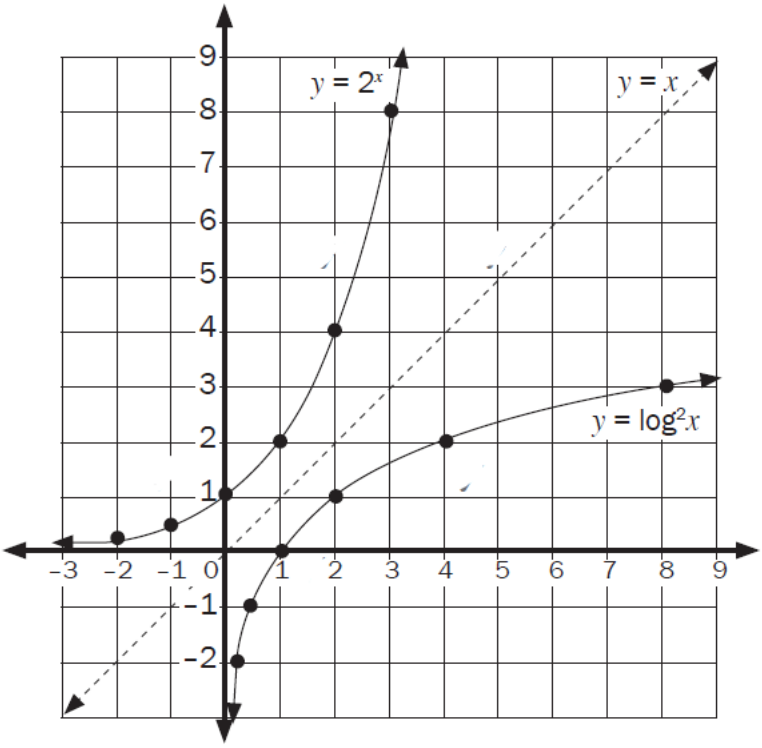
b) Sketch the graphs of $f(x)$, $f^{-1}(x)$ and $y = x$ on the same set of axes.

c) Write the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) The inverse of the exponential function $y = 2^x$ is $x = 2^y$ which can be written as $y = \log_2 x$.

b)



c) The domain and Range of $f(x)$

Domain: $x \in R$
Range: $y > 0$

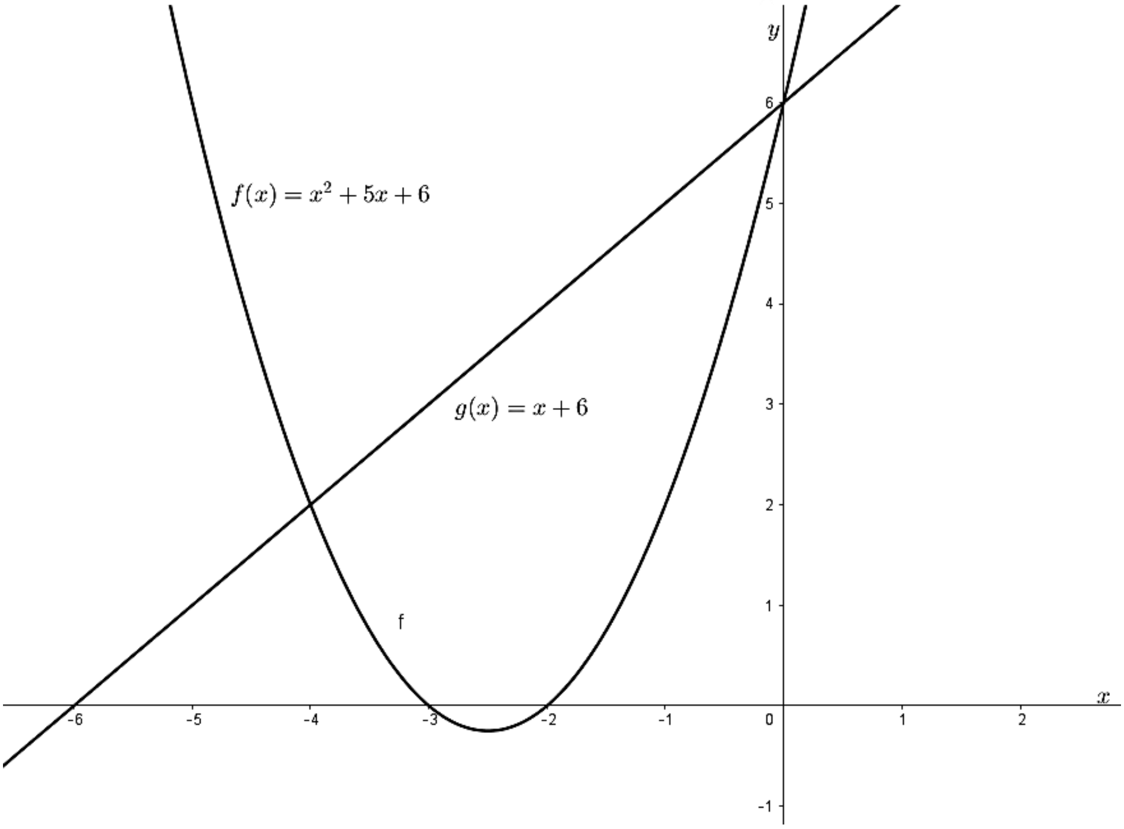
The domain and Range of $f^{-1}(x)$

Domain: $x > 0$
Range: $y \in 0$

SECTION 6: COMBINATIONS

Take note of the following when working with combination of functions:

Consider the graphs of f and g below

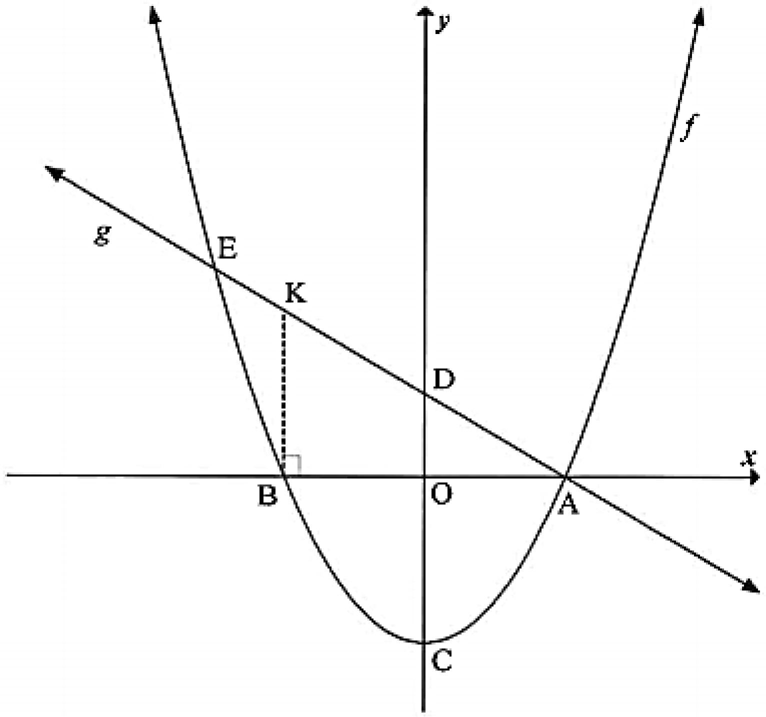


For $f(x) > 0$ or $f(x) < 0$	For $f(x) > g(x)$ or $f(x) < g(x)$	For $f(x).g(x) > 0$ or $f(x).g(x) < 0$
Focus on the $x - axis$. $f(x) > 0$ means where the graph of $f(x)$ is positive, which will be above the $x - axis$. And $f(x) < 0$ means where the graph of $f(x)$ is negative, which will be below the $x - axis$.	$f(x) > g(x)$ means where the graph of $f(x)$ is above the graph of $g(x)$. And $f(x) < g(x)$ means where the graph of $f(x)$ is below the graph of $g(x)$.	**$f(x).g(x) > 0$ means where the product of $f(x)$ and $g(x)$ is positive.** And **$f(x).g(x) < 0$ means where the product of $f(x)$ and $g(x)$ is negative.**

Example 1

QUESTION 5

The graphs of $f(x)=x^2-4$ and $g(x)=-x+2$ are sketched below. A and B are the x-intercepts of f. C and D are the y-intercepts of f and g respectively. K is a point on g such that $BK \parallel x$-axis. f and g intersect at A and E.



5.1 Write down the coordinates of C.

5.2 Write down the coordinates of D.

5.3 Determine the length of CD.

5.4 Calculate the coordinates of B.

5.5 Determine the coordinates of E, a point of intersection of f and g.

5.6 For which values of x will:

5.6.1 $f(x) < g(x)$

5.6.2 $f(x).g(x) \geq 0$

5.7 Calculate the length of AK.

Solutions

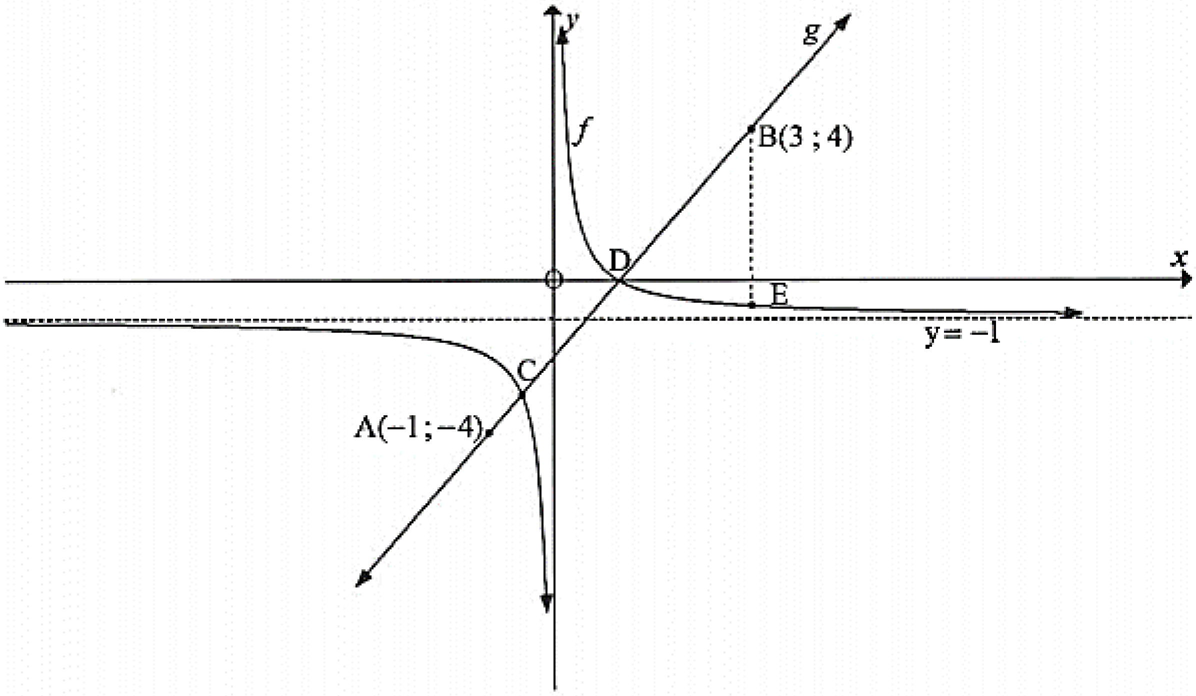
QUESTION 5	
5.1	C(0 ; –4)
5.2	D(0 ; 2)
5.3	CD = 2 – (–4) CD = 6 units/*eenhede*
5.4	$x^2 - 4 = 0$ $(x-2)(x+2) = 0$ $x = 2 \quad x = -2$ B(–2 ; 0)
5.5	$x^2 - 4 = -x + 2$ $x^2 + x - 6 = 0$ $(x-2)(x+3) = 0$ $x = 2 \quad x = -3$ E(–3 ; 5)
5.6.1	$-3 < x < 2$ **OR/*OF*** $(-3\ ;\ 2)$
5.6.2	$(-\infty\ ;\ -2] \cup \{2\}$
5.7	K(–2 ; 4) BK = 4 units/*eenhede* AB = 4 units/*eenhede* AK $= \sqrt{4^2 + 4^2}$ (Pythagoras) = 5,66 units/*eenhede*

Example 2

QUESTION 5

The sketch below shows f and g, the graphs of $f(x)=\frac{1}{x}-1$ and $g(x)=ax+q$ respectively.

Points A$(-1\,;-4)$ and B$(3\,;4)$ lie on the graph g.
The two graphs intersect at points C and D.
Line BE is drawn parallel to the y-axis, with E on f.



5.1 Show that $a=2$ and $q=-2$.

5.2 Determine the values of x for which $f(x)=g(x)$.

5.3 For what values of x is $g(x)\geq f(x)$?

5.4 Calculate the length of BE.

5.5 Write down an equation of h if $h(x)=f(x)+3$.

Solutions

5.1	$a = \text{gradient of } g$ $= \dfrac{-4-4}{-1-3}$ $= 2$ $4 = 2(3) + q$ $q = -2$ $g(x) = 2x - 2$
5.2	$\dfrac{1}{x} - 1 = 2x - 2$ $\dfrac{1}{x} = 2x - 1$ $1 = 2x^2 - x$ $2x^2 - x - 1 = 0$ $(2x+1)(x-1) = 0$ $x = -\dfrac{1}{2}$ or $x = 1$

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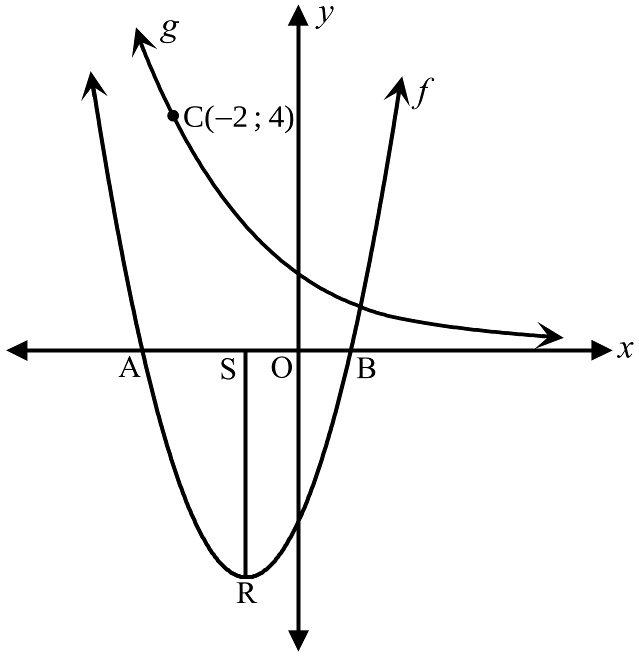
5.3	$-\frac{1}{2} \leq x < 0$ or/*of* $x \geq 1$ **OR/OF** $\left[-\frac{1}{2}\,;\,0\right) \cup [1\,;\,\infty)$
5.4	$f(3) = \frac{1}{3} - 1$ $= -\frac{2}{3}$ Length of BE $= 4 - f(3)$ $= 4 - \left(-\frac{2}{3}\right)$ $= 4 + \frac{2}{3}$ $= 4\frac{2}{3}$ **OR/OF** $\text{BE} = 2x - 2 - \frac{1}{x} + 1$ $= \frac{2x^2 - x - 1}{x}$ $(x = 3)\ \text{BE} = \frac{2(3)^2 - (3) - 1}{3}$ $= \frac{18 - 4}{3}$ $= 4\frac{2}{3}$
5.5	$h(x) = f(x) + 3$ $h(x) = \frac{1}{x} + 2$

Example 3

QUESTION 1

The graphs of $f(x)=2x^2+4x-6$ and $g(x)=a^x$ are represented in the sketch below. A and B are the x-intercepts of f and R is the turning point of f. The point $C(-2\,;4)$ is a point on the graph of g.



1.1 Show that $a=\frac{1}{2}$.

1.2 Determine the length of AB.

1.3 Determine the length of SR.

1.4 Write down the equation of h, if h is the reflection of f in the y-axis. Express your answer in the form $h(x)=a(x+p)^2+q$.

1.5 Write down the equation of g^{-1} in the form $y=....$

1.6 Sketch the graph of $y=g^{-1}(x)$ on a set of axes.

1.7 Determine the values of x for which:

1.7.1 $g^{-1}(x)\geq -2$

Solutions

1.1

$$y = a^x$$
$$\therefore 4 = a^{-2}$$
$$\therefore 4 = \frac{1}{a^2}$$
$$\therefore 4a^2 = 1$$
$$\therefore a^2 = \frac{1}{4}$$
$$\therefore a = \frac{1}{2}$$

1.2

$$0 = 2x^2 + 4x - 6$$
$$\therefore 0 = x^2 + 2x - 3$$
$$\therefore 0 = (x+3)(x-1)$$
$$\therefore x = -3 \quad \text{or} \quad x = 1$$
$$\therefore \text{AB} = 4 \text{ units}$$

1.3

$$x_{\text{R}} = -\frac{4}{2(2)} = -1$$
$$\therefore y_{\text{R}} = 2(-1)^2 + 4(-1) - 6 = -8$$
$$\therefore \text{SR} = 8 \text{ units}$$

Alternatively:

$$f'(x) = 4x + 4$$
$$\therefore 0 = 4x + 4$$
$$\therefore x = -1$$

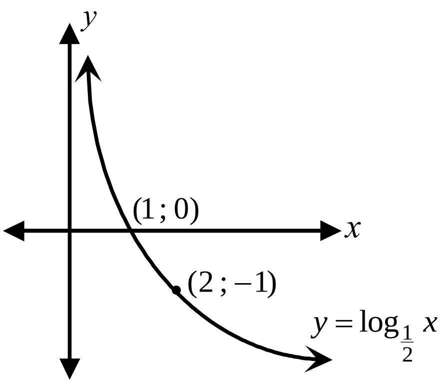
1.4

$$h(x) = 2(x-1)^2 - 8$$

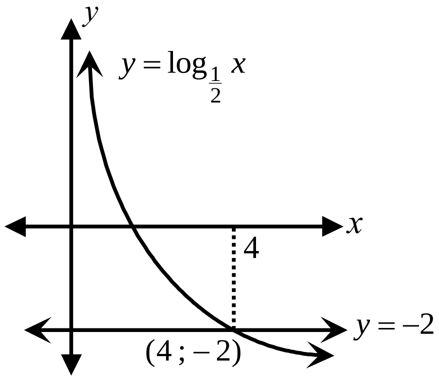
1.5

$$y = \left(\frac{1}{2}\right)^x$$
$$\therefore x = \left(\frac{1}{2}\right)^y$$
$$\therefore y = \log_{\frac{1}{2}} x$$

1.6



1.7.1



$$\log_{\frac{1}{2}} x = -2$$

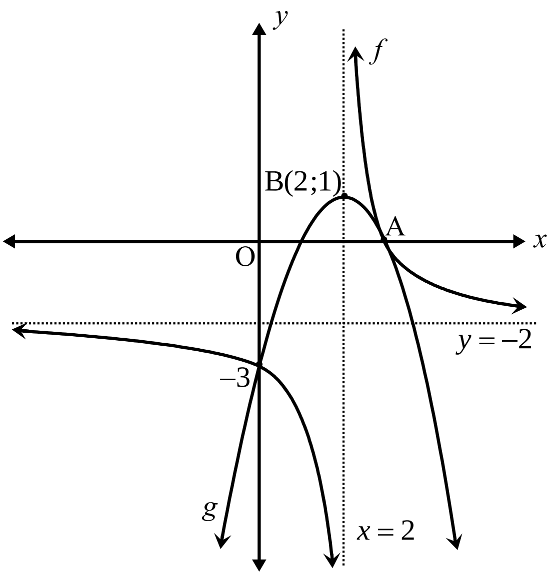
$$\therefore x = \left(\frac{1}{2}\right)^{-2} = 4$$

$$\therefore g^{-1}(x) \geq -2 \text{ for all } 0 < x \leq 4$$

Example 4

QUESTION 1

In the diagram below, the graph of $f(x)=\dfrac{a}{x+p}+q$ cuts the *y*-axis at -3 and the *x*-axis at A. The graph of $g(x)=m(x+n)^2+c$ intersects f at A, cuts the *y*-axis at -3 and has a turning point at $\text{B}(2;1)$.



Determine:

1.1 the equation of *f*.

1.2 the equation of *g*.

1.3 the length of OA.

1.4 the values of x for which $f(x).g(x) \leq 0.$

Solutions

1.1

$$y = \frac{a}{x-2} - 2$$

Substitute $(0\,;-3)$:

$$-3 = \frac{a}{0-2} - 2$$

$$\therefore -1 = \frac{a}{-2}$$

$$\therefore a = 2$$

$$\therefore f(x) = \frac{2}{x-2} - 2$$

1.2

$$y = m(x-2)^2 + 1$$

Substitute $(0\,;-3)$:

$$\therefore -3 = m(0-2)^2 + 1$$

$$\therefore -4 = m(-2)^2$$

$$\therefore -4 = 4m$$

$$\therefore m = -1$$

$$\therefore g(x) = -(x-2)^2 + 1$$

1.3

$$0 = \frac{2}{x-2} - 2$$

$$\therefore 0 = 2 - 2(x-2)$$

$$\therefore 0 = 2 - 2x + 4$$

$$\therefore 2x = 6$$

$$\therefore x = 3$$

$$\therefore \text{OA} = 3 \text{ units}$$

Alternatively:

$$0 = -(x-2)^2 + 1$$

$$\therefore (x-2)^2 = 1$$

$$\therefore x^2 - 4x + 4 = 1$$

$$\therefore x^2 - 4x + 3 = 0$$

$$\therefore (x-1)(x-3) = 0$$

$$x = 1 \quad \text{or} \quad x = 3$$

$$\therefore \text{OA} = 3 \text{ units}$$

1.4 $f(x).g(x) \leq 0$ for all

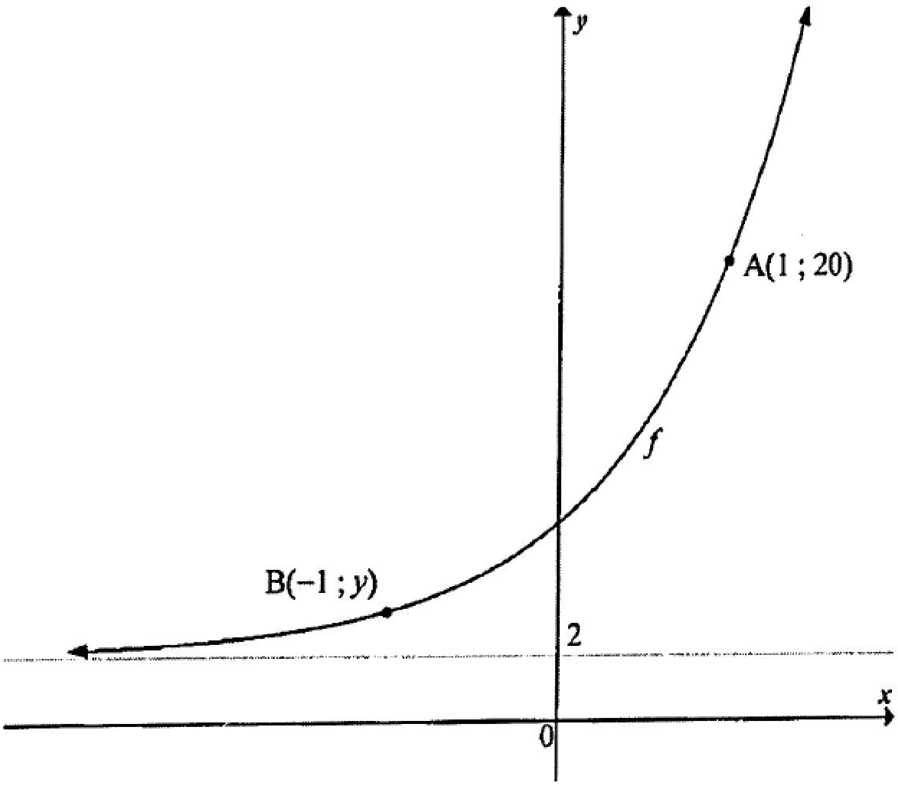
$$1 \leq x < 2 \quad \text{or} \quad x = 3$$

SECTION 7: THE AVERAGE GRADIENT BETWEEN TWO POINTS

The **average gradient** of a function between any two points is defined to be the **gradient of the line** joining the two points.

Example

The sketch below is the graph of $f(x) = 2.b^{x+1} + q$.
The graph of f passes through the points A(1 ; 20) and B (−1 ; y).
The line $y = 2$ is an asymptote of f.



1. Show that the equation of f is $f(x)=2(3)^{x+1}+2$

2. Calculate the y-coordinate of the point B.

3. Determine the average gradient of the curve between the points A and B.

Solutions

1. $$\begin{aligned} q &= 2 \\ f(x) &= 2.b^{x+1}+2 \\ 20 &= 2.b^{1+1}+2 \\ 18 &= 2.b^2 \\ 9 &= b^2 \\ b &= 3 \\ f(x) &= 2.3^{x+1}+2 \end{aligned}$$

2. $$\begin{aligned} y &= 2.3^{-1+1}+2 \\ y &= 2.1+2 \\ y &= 4 \end{aligned}$$

3. $$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{20-4}{1-(-1)} \\ &= 8 \end{aligned}$$

SECTION 8: ACTIVITIES OF FUNCTIONS AND GRAPHS

Activity 1

QUESTION 4

4.1 Given the function $p(x) = \left(\frac{1}{3}\right)^x$.

4.1.1 Is p an increasing or decreasing function? (1)

4.1.2 Determine p^{-1}, the inverse of p, in the form $y = \ldots$ (2)

4.1.3 Write down the domain of p^{-1}. (1)

4.1.4 Write down the equation of the asymptote of $p(x)-5$. (1)

4.2 Given: $f(x) = \dfrac{4}{x-1} + 2$

4.2.1 Write down the equations of the asymptotes of f. (2)

4.2.2 Calculate the x-intercept of f. (2)

4.2.3 Sketch the graph of f, label all asymptotes and indicate the intercepts with the axes. (4)

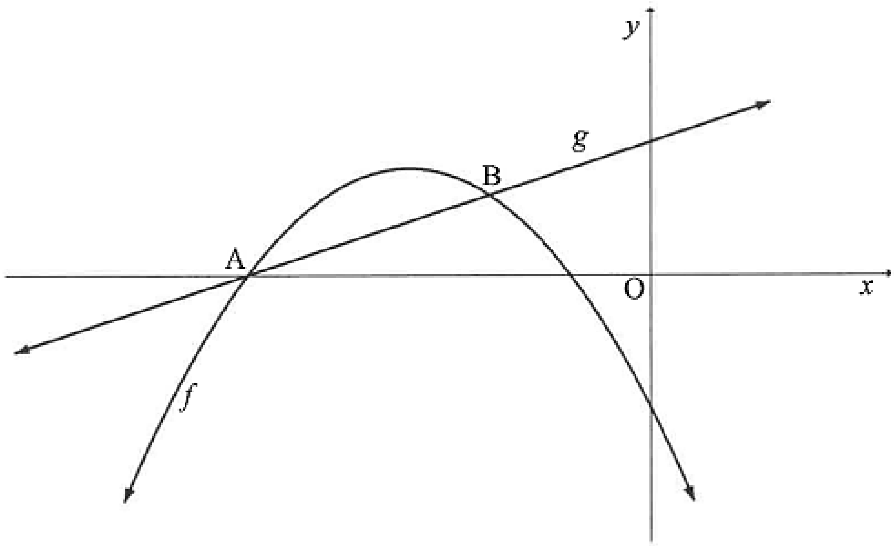
4.2.4 Use your graph to determine the values of x for which $\dfrac{4}{x-1} \geq -2$. (2)

4.2.5 Determine the equation of the axis of symmetry of $f(x-2)$, that has a negative gradient. (3)

[18]

QUESTION 5

The graphs of the functions $f(x) = -(x+3)^2 + 4$ and $g(x) = x + 5$ are drawn below.
The graphs intersect at A and B.



5.1	Write down the coordinates of the turning point of f.	(2)
5.2	Write down the range of f.	(1)
5.3	Show that the x-coordinates of A and B are -5 and -2 respectively.	(4)
5.4	Hence, determine the values of c for which the equation $-(x+c+3)^2 + 4 = (x+c) + 5$ has ONE negative and ONE positive root.	(2)
5.5	The maximum distance between f and g in the interval $x_A < x < x_B$ is k. If $h(x) = g(x) + k$, determine the equation of h in the form $h(x) = ...$	(5)
		[14]

Activity 2

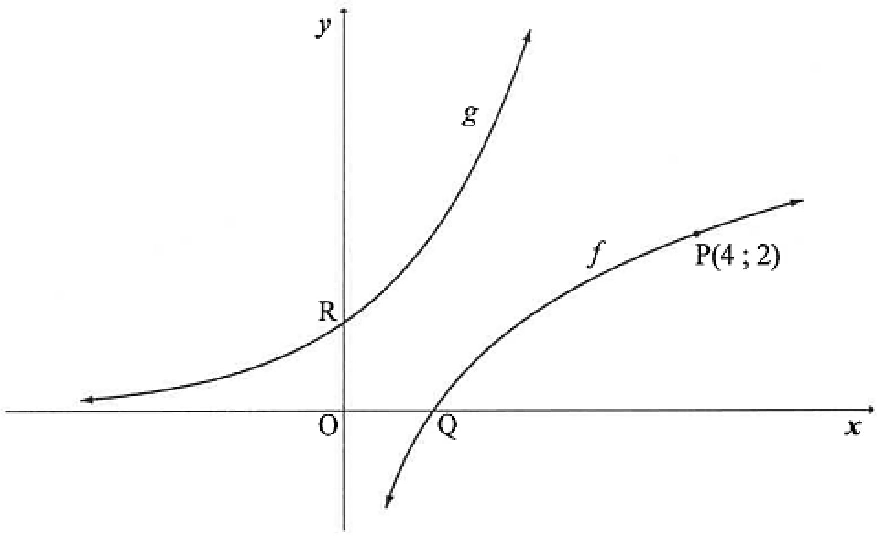
QUESTION 4

Given: $g(x) = \dfrac{1}{x-1} + 2$

4.1	Write down the equations of the asymptotes of g.	(2)
4.2	Draw a graph of g, indicating any intercepts with the axes and asymptotes.	(4)
4.3	Determine the values of x where $g(x) > 0$.	(2)
4.4	Determine the equation of the axis of symmetry of g which has a negative gradient.	(2)
		[10]

QUESTION 5

In the diagram, the graphs of $f(x) = \log_a x$ and g are drawn. Graph g is the reflection of f in the line $y = x$. Graph f passes through the point P(4 ; 2). Q is the x-intercept of f and R is the y-intercept of g.



5.1 Write down the coordinates of P$'$, the image of P on g. (2)

5.2 Show that $a = 2$. (2)

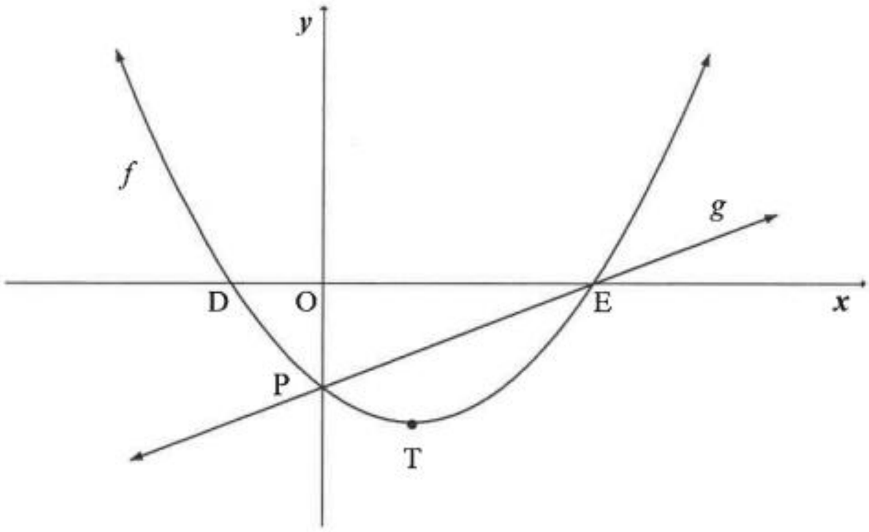
5.3 Write down the equation of g in the form $y = \ldots$ (1)

5.4 T is a point on f in the first quadrant where TR is parallel to the x-axis. Calculate the area of ΔRTP$'$. (4)

[9]

QUESTION 6

The graphs of $f(x) = x^2 - 2x - 3$ and $g(x) = mx + c$ are drawn below. D and E are the x-intercepts and P is the y-intercept of f. The turning point of f is T$(1 ; -4)$. The graphs of f and g intersect at P and E.



6.1 Write down the range of f. (1)

6.2 Calculate the coordinates of D and E. (3)

6.3 Determine the equation of g. (2)

6.4 Write down the values of x for which $f(x) - g(x) > 0$. (2)

6.5 Determine the maximum vertical distance between h and g if $h(x) = -f(x)$ for $x \in [-2 ; 3]$. (5)

6.6 Given: $k(x) = g(x) - n$.

Determine n if k is a tangent to f. (5)

[18]

Activity 3

QUESTION 4

Given: $f(x) = a^x - 1$ for $a > 0$. $B\left(2\,;\dfrac{-5}{9}\right)$ is a point on f.

4.1 Calculate the value of a. (2)

4.2 Write down the range of f. (1)

4.3 Sketch the graph of f. Clearly show the intercepts with the axes and asymptotes, if any. (3)

4.4 It is further given that C is a point on f at $y = \dfrac{19}{8}$.

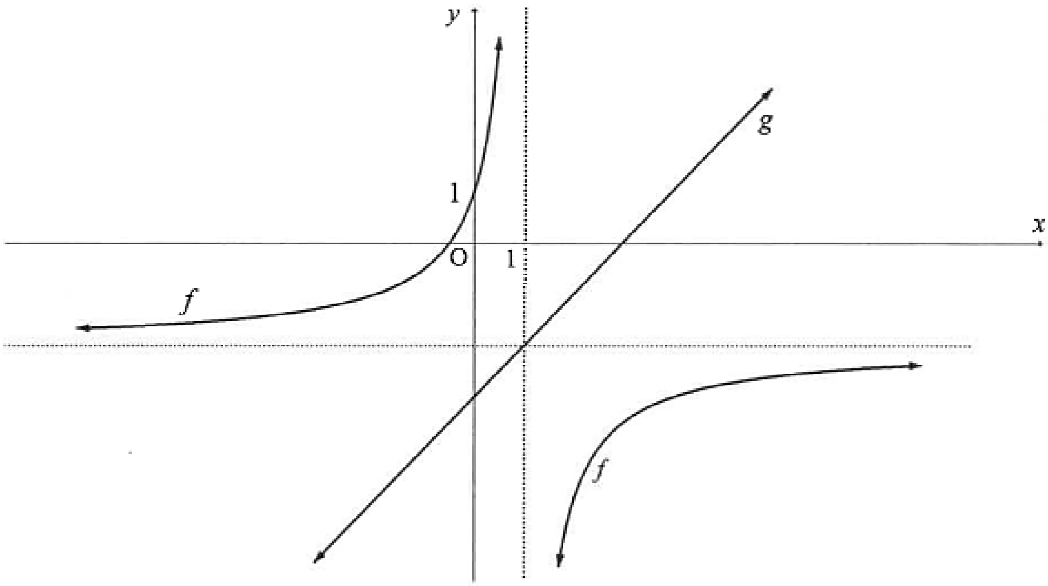
Determine the coordinates of $C^{/}$, the image of C, when C is reflected about the line $y = x$. (3)

[9]

QUESTION 5

Sketched below is the graph of $f(x) = \dfrac{a}{x+p} + q$ having the domain $(-\infty\,;1) \cup (1\,;\infty)$.

The graph of f cuts the y-axis at $(0\,;1)$. A line of symmetry of f is given by $g(x) = x - 3$.



5.1	Write down the value of p.	(1)
5.2	Determine the equation of the horizontal asymptote of f.	(2)
5.3	Calculate the value of a.	(2)
5.4	For which values of x is $f(x) \geq 0$?	(3)

5.5 Graph f undergoes a transformation to h where:

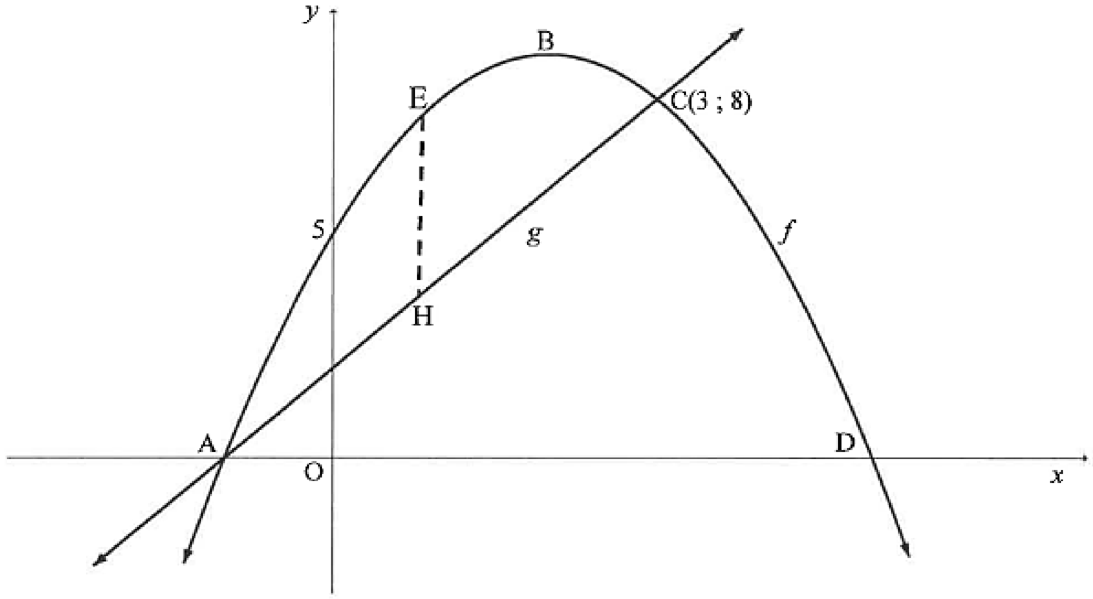
- The domain and range of h are the same as that of f
- $h'(x)$, the derivative of h, is negative on its domain

Describe a possible transformation that f could have undergone to result in h. (2)

[10]

QUESTION 6

In the diagram below, the graphs of $f(x) = -x^2 + 4x + 5$ and g, a straight line, are drawn. $C(3\,;8)$ is a point of intersection of f and g. EH is drawn parallel to the y-axis, with E a point on f and H a point on g.



6.1 Calculate the coordinates of B, the turning point of f. (3)

6.2 Show that the equation of the line through A and C is given by $g(x) = 2x + 2$. (3)

6.3 Calculate the maximum length of EH for $f > g$. (4)

6.4 Given: $k(x) = f(x+m) = -x^2 - 2mx - m^2 + 4x + 4m + 5$

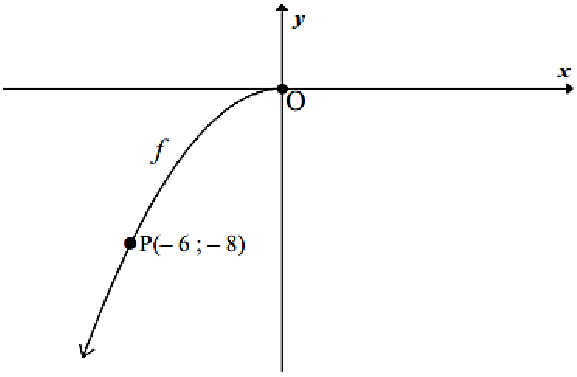
Determine the value of m such that g is a tangent to k. (5)

[15]

Activity 4

QUESTION 6

The graph of $f(x) = ax^2$, $x \leq 0$ is sketched below.
The point P(– 6 ; – 8) lies on the graph of *f*.



6.1 Calculate the value of a. (2)

6.2 Determine the equation of f^{-1}, in the form $y = \ldots$ (3)

6.3 Write down the range of f^{-1}. (1)

6.4 Draw the graph of f^{-1} **ON YOUR BOOK** . Indicate the coordinates of a point on the graph different from (0 ; 0). (2)

6.5 The graph of f is reflected across the line $y = x$ and thereafter it is reflected across the x-axis. Determine the equation of the new function in the form $y = \ldots$ (3)

[11]

DIFFERENTIAL CALCULUS

Overview:

1. The equations of tangents to graphs.
2. The ability to sketch graphs of cubic functions.
3. Practical problems involving optimization and rates of change (including the calculus of motion).

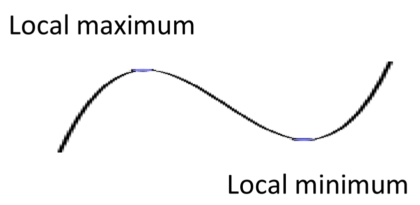
(SOURCE: CURRICULUM AND ASSESSMENT POLICY STATEMENT(CAPS)SENIOR PHASE GRADES (10 – 12) MATHEMATICS)

SECTION 1: SKETCHING OF THE CUBIC FUNCTION

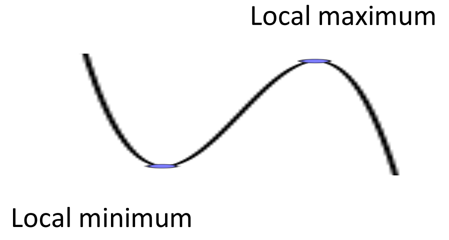
Take note of the standard form and shape of a cubic function:

Standard form: $y = ax^3 + bx^2 + cx + d$

If $a > 0$



If $a < 0$



Steps to follow when sketching: ***(N.B sometimes, if not most of the time, you will be directed by the question as to where to start,.. see the worked example)***

1. Determine the x and y - intercepts
    a. $x-$intercept(s)
        i. Let y = 0 and simplify
        ii. Find the factor
        iii. Solve for x
    b. $y-$intercept
        i. Let x = 0 and solve for y
2. Determine the turning points
    a. Find the first derivative
    b. Equate to zero
    c. Solve for x (these are x -values of the turning points)
    d. Substitute the x -values into the original function to find the corresponding y-values)
3. Sketch

N.B Take note of different ways of finding the point of inflection:

Way 1

1. Find the second derivative of the function. E.g $f''(x)$
2. Equate the second derivative to zero and solve for x. You now have x-value of point of inflection.
3. Substitute the x-value into the original function, e.g $f(x)$, to get the corresponding y-value, and thus the point of inflection.

Way 2 (Using x-values the turning points: The point of inflection lies half-way between the turning points of a cubic function)

1. Find the x-value of the point of inflection by adding together x-values of the turning points and then dividing the result by 2. You now have the x-value of the point of inflection.
2. Substitute the x-value into the original function, e.g $f(x)$, to get the corresponding y-value, and thus the point of inflection.

Worked example(s)

Example 1

Given: $f(x) = x^3 + 2x^2 - 7x + 4$

1 Show that $(x-1)$ is a factor of $f(x)$.

2 Hence, or otherwise, find the x-intercepts of f.

3 Determine the coordinates of the turning points of f.

4 Sketch the graph of f on the ANSWER SHEET provided. Clearly show ALL the intercepts with the axes and the turning points.

Solutions

1.

$$f(x) = x^3 + 2x^2 - 7x + 4$$
$$f(1) = (1)^3 + 2(1)^2 - 7(1) + 4$$
$$\therefore f(1) = 0$$
$$\therefore x - 1 \text{ is a factor of } f$$

2.

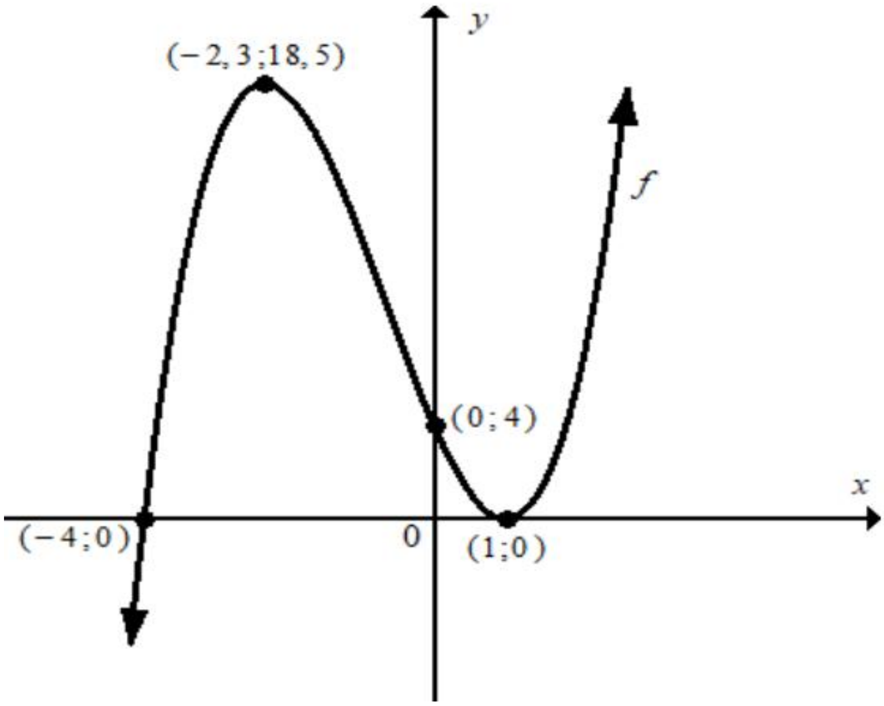
x-intercepts:

$$f(x) = 0$$
$$x^3 + 2x^2 - 7x + 4 = 0$$
$$(x-1)(x^2 + 3x - 4) = 0$$
$$(x-1)(x-1)(x+4) = 0$$
$$x = 1 \text{ or } x = -4$$

3.

$$f(x) = x^3 + 2x^2 - 7x + 4$$
$$f^{/}(x) = 3x^2 + 4x - 7$$
$$f^{/}(x) = 0$$
$$\therefore 3x^2 + 4x - 7 = 0$$
$$(3x + 7)(x - 1) = 0$$
$$\therefore x = -\frac{7}{3} \text{ or } x = 1$$
$$(-2,3\,;18{,}5) \text{ and } (1;0)$$

4.



Example 2

Given: $f(x)=x^3-x^2-8x+12$

1.1.Draw a sketch graph of f, showing all intercepts with axes, and turning points.

Solution

***y*-intercept**:

$f(0)=0^3-0^2-8\cdot 0+12$

$=12$

y -intercept is given by $(0;12)$

***x*-intercept:**

$(x-2)(x^2+x-6)=0$

(x – 2)(x + 3)(x – 2) = 0

$(x-2)=0 \quad or \quad (x+3)=0 \quad or \quad (x-2)=0$

$x=2 \quad or \quad x=-3 \quad or \quad x=2$

$x-$ intercepts are given by $(2;0)$ and $(-3,0)$

Turning points:

$f'(x)=0$

$3x^2-2x-8=0$

$(3x+4)(x-2)=0$

$x=-\frac{4}{3}$ or $x=2$

$y=(-\frac{4}{3})^3-(-\frac{4}{2})^2-8(-\frac{4}{3})+12$

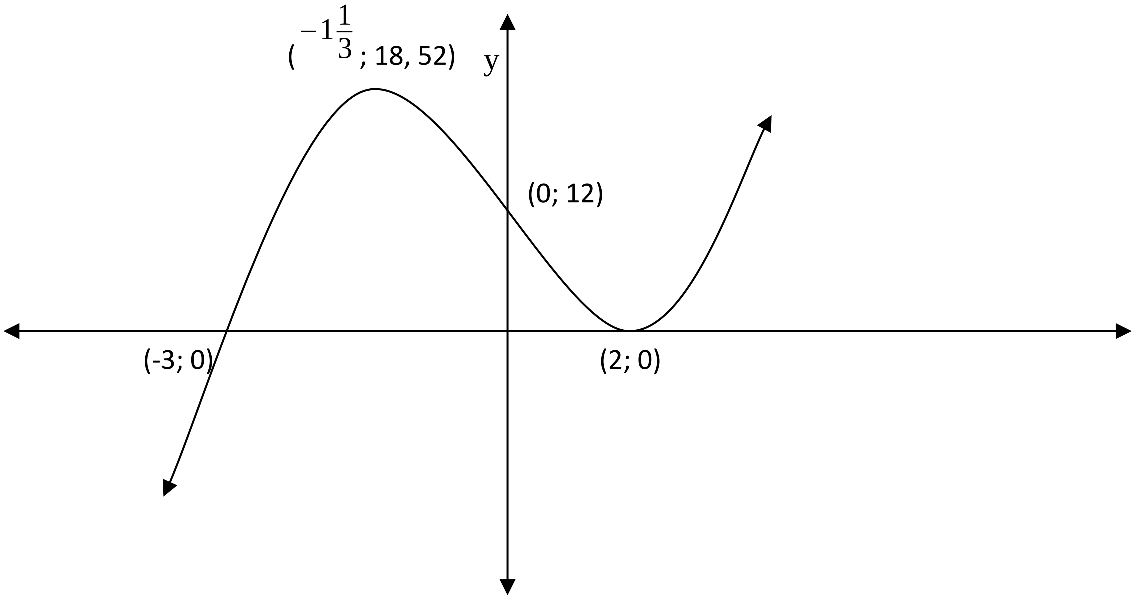
$=\frac{500}{27}$

$=18{,}52$

$= \frac{500}{27}= 18{,}52$

Local Maximum: $\left(-1\frac{1}{3}; 18,52\right)$

Local Minimum: (2; 0)



1.2 Calculate the point of inflection.

Solution

$f''(x) = 6x - 2$

$6x - 2 = 0$

$x = \frac{1}{3}$

SECTION 2: FINDING EQUATION OF CUBIC FUNCTION AND EQUATION OF A TANGENT

Important terminology and/or notes

- Finding the parameters of the function leads to finding the equation of that function
- Equation of tangent is the equation of a straight line: $y = mx + c$
    - What you need…
        - The point of contact
        - The gradient

Example 1

Determine the equation of the tangent to the curve defined by $g(x) = -x^2 - x$ at the point where $x = 2$.

Solution

$g(x) = -x^2 - x$

$g(2) = -(2)^2 - 2 = -6$

The point of contact is $(2; -6)$

$g'(x) = -2x - 1$

$\therefore m_{\tan} = g'(2) = -2(2) - 1 = -5$

$y = mx + c$ OR $y - y_1 = m(x - x_1)$

$-6 = -5(2) + c$ OR $y - (-6) = -5(x - 2)$

$c = 4$ OR $y + 6 = -5x + 10$

$\therefore y = -5x + 4$

Example 2

Given: $h(x) = -x^3 + ax^2 + bx$ and $g(x) = -12x$. P and Q(2 ; 10) are the turning points of h. The graph of h passes through the origin.

Show that $a = \frac{3}{2}$ and $b = 6$.

Solution

Substitute Q(2; 10) into

$$h(x) = -x^3 + ax^2 + bx$$

$$-2^3 + a(2^2) + b(2) = 10$$

$$-8 + 4a + 2b = 10$$

$$2a + b = 9 \quad \text{..................line 1}$$

$$h'(x) = -3x^2 + 2ax + b$$

$$\text{At Q}: \quad h'(2) = 0$$

$$-3(2)^2 + 2a(2) + b = 0$$

$$-12 + 4a + b = 0$$

$$4a + b = 12 \quad \text{.............line 2}$$

$$\text{line 2} - \text{line 1}: \quad 2a = 3$$

$$a = \frac{3}{2}$$

$$\text{Substitute in line 1}: \quad b = 6$$

SECTION 3: OPTIMISATION AND RATE OF CHANGE, INCLUDING CALCULUS OF MOTION

Important terminology and/or notes

Maximum and minimum occur at the turning points, thus when maximizing or minimizing, find the first derivative, equate the derivative to zero and solve for the unknown, x in most cases. The value(s) of x found after solving is/are where the maximum or the minimum occurs. Check the value(s) whether they give maximum or minimum values by substituting into original equation.

Worked example(s)

Example (A)

An industrial open water tank, as shown in the picture below, has an inlet pipe and an outlet pipe. The depth of the water in the tank changes continually.

The equation $D(t) = 4 + 0,5t^2 - 0,25t^3$ gives the depth (in metres) of the water, where t represents the time (in hours) that has lapsed since the depth reading was taken at 09:00.

Determine:

1 The depth of the water in the tank at 11:00

2 The rate of change of the depth of the water in the tank at 12:00

Solutions

1.
After 2 hrs
$$D(2) = 4 + 0,5(2)^2 - 0,25(2)^3 \text{ m}$$
$$= 4m$$

2.
$$D = 4 + 0,5t^2 - 0,25t^3$$
$$D'(t) = t - 0,75t^2$$
At 12:00 (3 hours later):
$$D'(3) = (3) - 0,75(3)^2$$
$$= -3,75\, m.h^{-1}$$

Example (B)

The profit (in R1000s) yielded by a company, using a machine that produces bottle caps, is dependent on the average speed at which the machine runs.

The profit (P) is calculated using the formula:

$P = -3v^2 + 30v$,

where v is the average speed (in kilometres per hour) and $v > 0$.

1 Calculate the average speed at which neither a profit, nor a loss is yielded.

2 Determine at what average speed the machine should run so that a maximum profit will be obtained.

3 Hence, or otherwise, calculate the resulting maximum profit.

Solutions

1.

$$P = -3v^2 + 30v$$
Neither profit nor loss at $P = 0$
$$-3v^2 + 30v = 0$$
$$-3v(v-10) = 0$$
$$\therefore v = 0 \text{ or } v = 10$$
$$v = 10\,km.h^{-1}$$

2.

$$P = -3v^2 + 30v$$
$$\frac{dP}{dv} = -6v + 30 = 0$$
$$\therefore v = 5\,km.h^{-1}$$

3.

$$P_{max}\text{(in R1000)} = -3(5)^2 + 30(5) = 75$$
OR R75 000

Example (C)

A particle moves along a straight line. The distance, s, (in metres) of the particle from a fi point on the line at time t seconds ($t \geq 0$) is given by $s(t) = 2t^2 - 18t + 45$.

10.1 Calculate the particle's initial velocity. (Velocity is the rate of change of distance.)

10.2 Determine the rate at which the velocity of the particle is changing at t seconds.

10.3 After how many seconds will the particle be closest to the fixed point?

Solution

10.1	$s(t)=2t^2-18t+45$ $s'(t)=4t-18$ $s'(0)=4(0)-18$ $=-18\ m/s$
10.2	$s''(t)=4\ \text{m/s}^2$
10.3	$4t-18=0$ $4t=18$ $t=\frac{9}{2}\text{seconds or } 4{,}5\text{seconds}$ **OR** $s(t)=2\left(t-\frac{9}{2}\right)^2+\frac{9}{2}$ $t=\frac{9}{2}\text{seconds or } 4{,}5\text{seconds}$ **OR** $s(t)=2t^2-18t+45$ $t=-\frac{-18}{2(2)}$ $t=\frac{9}{2}\text{seconds or } 4{,}5\text{seconds}$

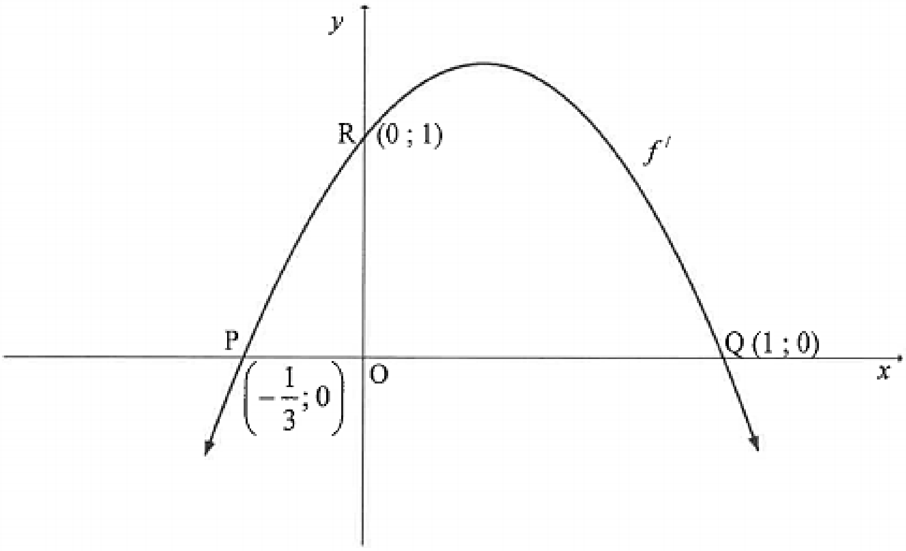
SECTION 4: ACTIVITIES OF DIFFERENTIAL CALCULUS

Activity 1

QUESTION 8

The graph of $y = f'(x) = mx^2 + nx + k$ is drawn below.

The graph passes the points $P\left(-\frac{1}{3}\,;0\right)$, $Q(1\,;0)$ and $R(0\,;1)$.



8.1 Determine the values of m, n and k. (6)

8.2 If it is further given that $f(x) = -x^3 + x^2 + x + 2$:

8.2.1 Determine the coordinates of the turning points of f. (3)

8.2.2 Draw the graph of f. Indicate on your graph the coordinates of the turning points and the intercepts with the axes. (5)

8.3 Points E and W are two variable points on f' and are on the same horizontal line.

- h is a tangent to f' at E.
- g is a tangent to f' at W.
- h and g intersect at $D(a\,;b)$.

8.3.1 Write down the value of a. (1)

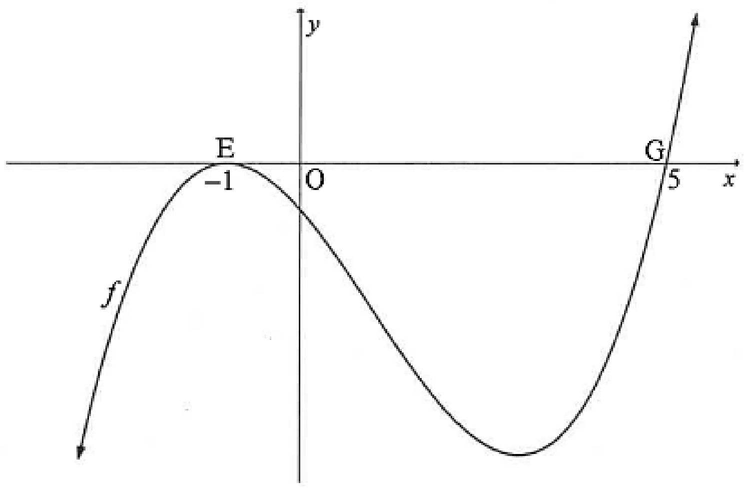
8.3.2 Determine the value(s) of b for which h and g will no longer be tangents to f'. (2)

[17]

Activity 2

QUESTION 9

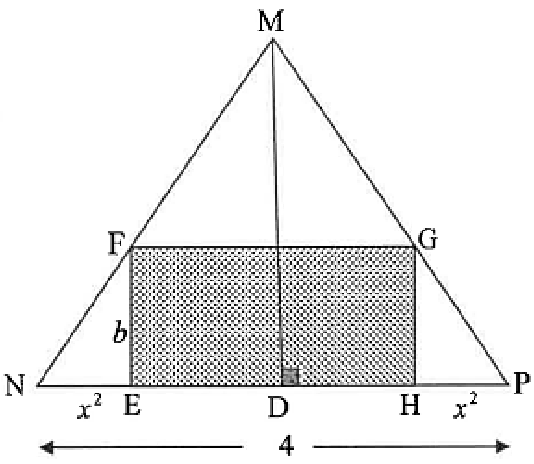
The graph of $f(x) = ax^3 + bx^2 + cx - 5$ is drawn below. E(– 1 ; 0) and G(5 ; 0) are the x-intercepts of f.



9.1	Show that $a = 1$, $b = -3$ and $c = -9$.	(3)
9.2	Calculate the value of x for which f has a local minimum value.	(4)
9.3	Use the graph to determine the values of x for which $f''(x).f(x) > 0$.	(3)
9.4	For which values of t will the graph of $p(x) = f(x) + t$ have two distinct positive roots and one negative root?	(3)
		[13

QUESTION 10

EHGF is a rectangle. HE is produced x^2 cm to N and EH is produced x^2 cm to P. NF produced intersects PG produced at M to form an isosceles triangle MNP with NM = MP. D lies on NP where MD $\perp$ NP. NP = 4 cm and MD = 3 cm.



10.1 Show that the area of EFGH is given by $A(x) = 6x^2 - 3x^4$.

10.2 Calculate the maximum area of rectangle EFGH.

Activity 3

QUESTION 8

Given: $f(x) = -x^3 + 6x^2 - 9x + 4 = (x-1)^2(-x+4)$

8.1 Determine the coordinates of the turning points of f. (4)

8.2 Draw a sketch graph of f. Clearly label all the intercepts with the axes and any turning points. (4)

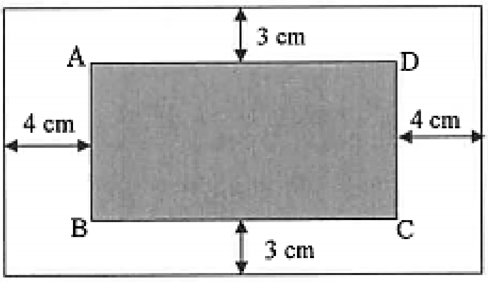
8.3 Use the graph to determine the value(s) of k for which $-x^3 + 6x^2 - 9x + 4 = k$ will have three real and unequal roots. (2)

8.4 The line $g(x) = ax + b$ is the tangent to f at the point of inflection of f. Determine the equation of g. (6)

8.5 Calculate the value of θ, the acute angle formed between g and the x-axis in the first quadrant. (2)
[18]

QUESTION 9

The diagram below represents a printed poster. Rectangle ABCD is the part on which the text is printed. This shaded area ABCD is 432 cm^2 and $AD = x$ cm. ABCD is 4 cm from the left and right edges of the page and 3 cm from the top and bottom of the page.



9.1 Show that the total area of the page is given by:

$$A(x) = \frac{3\,456}{x} + 6x + 480$$
(3)

9.2 Determine the value of x such that the total area of the page is a minimum. (3)
[6]

Activity 4

QUESTION 8

Given: $f(x)=x^3+4x^2-7x-10$

8.1	Write down the y-intercept of f.	(1)
8.2	Show that 2 is a root of the equation $f(x)=0$.	(2)
8.3	Hence, factorise $f(x)$ completely.	(3)
8.4	If it is further given that the coordinates of the turning points are approximately at $(0{,}7\,;-12{,}6)$ and $(-3{,}4\,;20{,}8)$, draw a sketch graph of f and label all intercepts and turning points.	(3)
8.5	Use your graph to determine the values of x for which:	
8.5.1	$f'(x)<0$	(2)
8.5.2	The gradient of a tangent to f will be a minimum	(2)
8.5.3	$f'(x).f''(x)\leq 0$	(3)
		[16]

QUESTION 9

A wire, 12 metres long, is cut into two pieces. One part is bent to form an equilateral triangle and the other a square. A side of the triangle has a length of $2x$ metres.

9.1	Write down the length of a side of the square in terms of x.	(2)
9.2	If this square is now used as the base of a rectangular prism with a height of $4x$ metres, determine the maximum volume of the rectangular prism.	(7)
		[9]

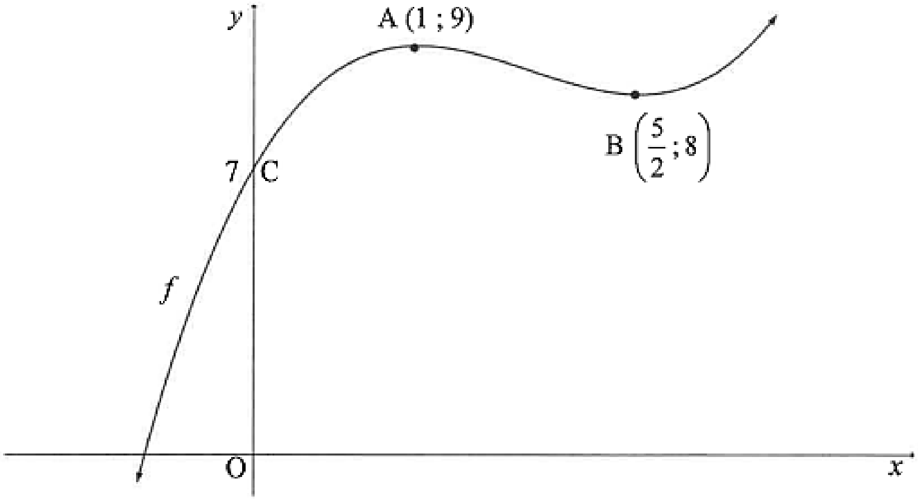
Activity 5

8.2	Determine the equation of the tangent to $f(x)=x^3-4x^2+2x+3$ at $x=2$.	(3)
8.3	Given: $f(x)=-6x^2$	
8.3.1	Determine $f'(x)$ from first principles.	(5)
8.3.2	Write down how you will restrict the domain of f such that f^{-1}, the inverse of f, is a function.	(1)
8.3.3	Determine the equation of f^{-1} for $f^{-1}(x)\leq 0$. Write your answer in the form $y=\ldots$	(3)
		[18]

QUESTION 9

A$(1\,;9)$ and B$\left(\frac{5}{2};8\right)$ are the turning points of graph f below.

C$(0\,;7)$ is the y-intercept of f.



9.1	For which values of x is f decreasing?	(2)
9.2	Write down the x-intercepts of f', the derivative of f.	(2)
9.3	For which values of x will f be concave up?	(2)
9.4	Determine the value of k for which $y = f(x) + k$ will have THREE positive x-intercepts.	(2)
		[8]

QUESTION 10

A cyclist rode from town P and stopped at town T. The speed (in km/h) at which this cyclist rode, is represented by the equation $s'(t) = -3t^2 + 18t$.

NOTE: Speed is the rate of change in distance with respect to time.

10.1	Calculate the maximum speed that the cyclist reached on this ride.	(3)
10.2	Calculate the distance between town P and town T.	(5)
		[8]

PROBABILITY

Overview:

(a) Dependent and independent events.

(b) Venn diagrams or contingency tables and tree diagrams as aids to solving probability problems (where events are not necessarily

(SOURCE: CURRICULUM AND ASSESSMENT POLICY STATEMENT(CAPS)SENIOR PHASE GRADES (10 – 12) MATHEMATICS)

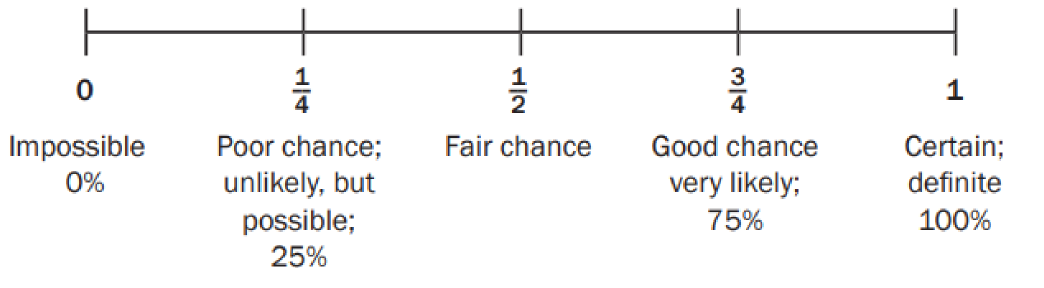
SECTION 1: GENERAL PROBABILITY

Probability refers to the likelihood or chance of an event taking place.

$$\textbf{\textit{The probability of an event}} = \frac{\textbf{\textit{number of favourable outcomes}}}{\textbf{\textit{total number of outcomes}}}$$

This ratio can be expressed as a common fraction, a decimal fraction or a percentage. So a out of 8 can be written as $\frac{5}{8}$ or as 0,625 or as 62,5%.

We can use a **probability scale** to decide what chance there is of an event happening.



Notations that are used are to find probability of an event:

- $P(A)$ means the probability of event A occurring
- $P(A')$ or $P(not\ A)$ means the probability of event A not occurring.
- $P(A\ or\ B) = P(A \cup B)$ means the probability of A or B occurring.

  $\cup$ is the symbol for **or**, it is also known as union.
- $P\ (A\ and\ B) = P\ (\ A \cap B\ )$ means the probability of A and B occurring.

  $\cap$ is the symbol for **and**, it is also known as intersection.

The Identity

For any two events A and B:

$$\boldsymbol{P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)}$$

The Addition Rule for Mutually Exclusive Events

Two events are called mutually exclusive if both events cannot take place at the same time.

When events A and B are mutually exclusive, $\boldsymbol{P(A\ and\ B) = 0}$

then $\boldsymbol{P(A\ or\ B) = P(A) + P(B)}$

The Complementary Rule:

For two events A and B:

$$\boldsymbol{P(not\ A\ ) = 1 - P(A)}$$

The Product Rule for Independent Events

When two events A and B are independent, then

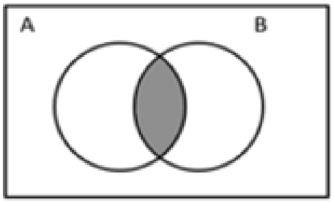
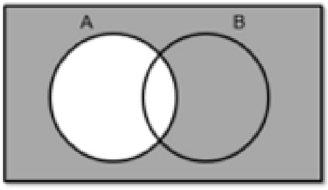
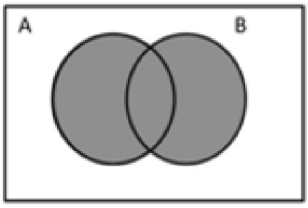
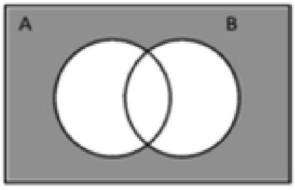
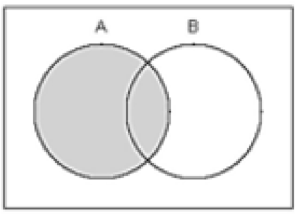
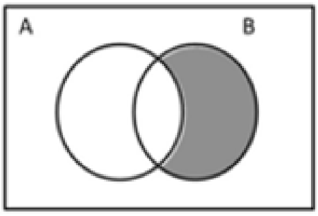
$$P(A \text{ and } B) = P(A) \times P(B)$$

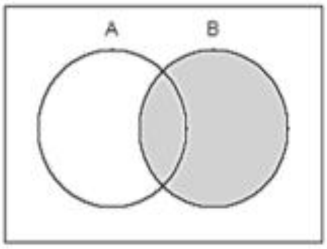
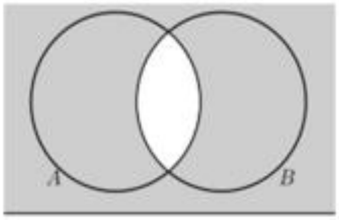
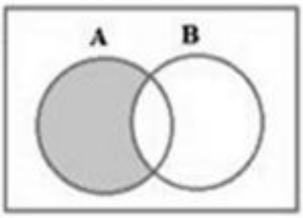
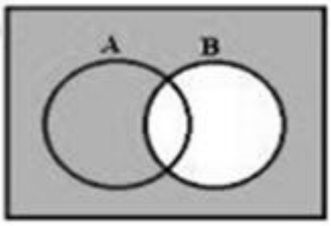
WARNING

Just because two events are mutually exclusive does not necessarily mean that they are independent. To test whether events are mutually exclusive, always check that $P(A \text{ and } B) = 0$. To test whether events are independent, always check that $P(A \text{ and } B) = P(A) \times P(B)$. See the exercises below for examples of events that are mutually exclusive and independent in different combinations.

SECTION 2: VENN DIAGRAMS

The shaded regions represent the events above the Venn diagrams.

A and B	not A
	
A or B	not (A or B)
	
A	B only
	

B	not (A and B)
	
A only	not B
	

Worked Example 1

QUESTION 4

4.1 A survey of 80 students at a local library indicated the reading preferences below:

44 read the *National Geographic* magazine

33 read the *Getaway* magazine

39 read the *Leadership* magazine

23 read both *National Geographic* and *Leadership* magazines

19 read both *Getaway* and *Leadership* magazines

9 read all three magazines

69 read at least one magazine

4.1.1 How many students did not read any magazine?

4.1.2 Let the number of students who read *National Geographic* and *Geta* but not *Leadership*, be represented by x. Draw a Venn diagra represent reading preferences.

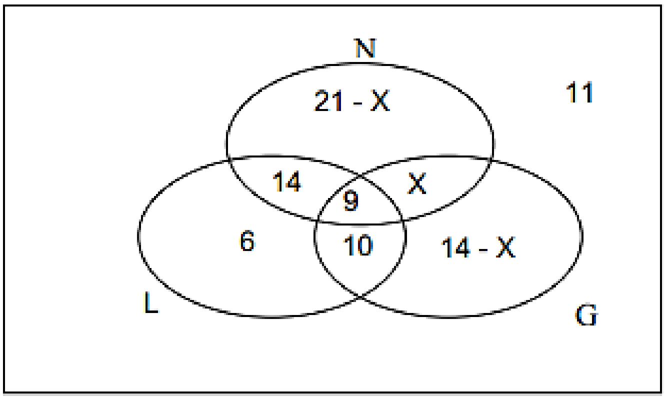
4.1.3 Hence show that $x = 5$.

4.1.4 What is the probability, correct to THREE decimal places, that a stu selected at random will read at least two of the three magazines?

Solution

4.1.1 11 students

4.1.2 Let N represent students reading the *National Geographic* magazine, G represent students reading the *Getaway* magazine and L represent students reading the *Leadership* magazine.



4.1.3 $21 - x + x + 14 - x + 9 + 14 + 10 + 6 + 11 = 80$

$$85 - x = 80$$
$$x = 5$$

4.1.4 P(student reads at least two magazines) $= \dfrac{5+14+10+9}{80} = 0{,}475$

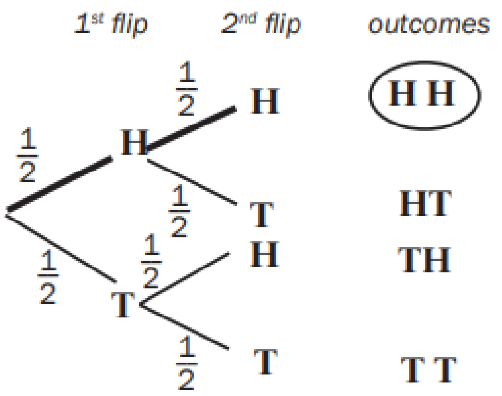
4.2.1

P(smoke detected by device A or device B)
= P(smoke detected by A) + P(smoke detected by B) – P(smoke detected by both)
$= 0{,}95 + 0{,}98 - 0{,}94$
$= 0{,}99$

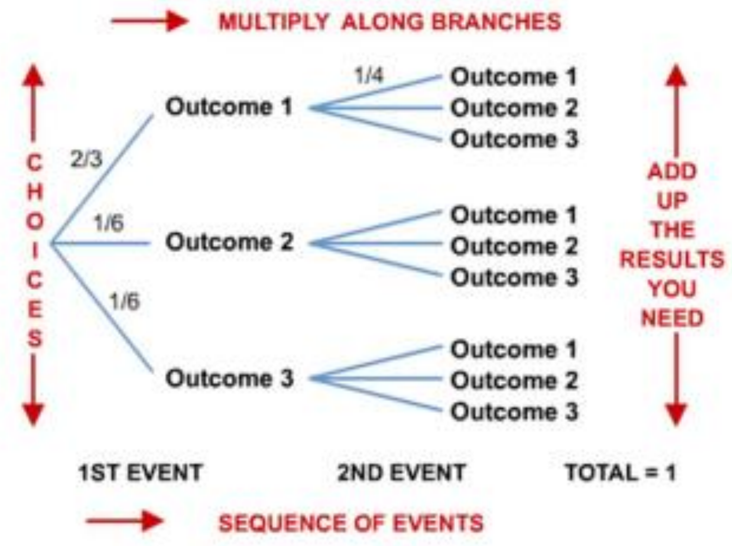
4.2.2 P(smoke not detected) $= 1 - 0{,}99 = 0{,}01$

SECTION 3: TREE DIAGRAMS

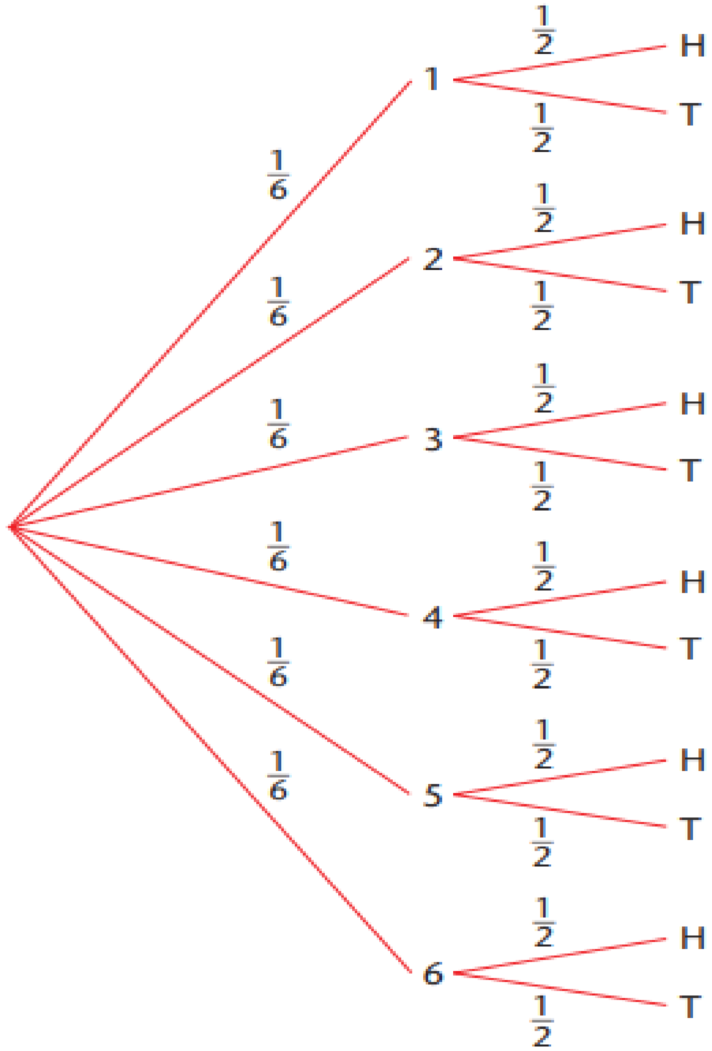
A tree diagram is a picture that helps you to list all possible outcomes of the events. Here is the tree diagram for P(H and H) if you flip a coin twice:



In a tree diagram (once it is complete), to find the probability of an outcome we multiply across the branches. If more than one outcome matches the results required, then addition of the answers gained from the multiplication is used. Remember that each 'clump' of branches should add up to a probability of 1.



When more than one event takes place consecutively or simultaneously, it is useful to represent them as a tree diagram. We represent each event by a column of branches, and the number of branches is determined by the number of possible outcomes for that event. For example, if a die is thrown, there are six possible outcomes, numbers 1 to 6, which we represent by six different line (branches) drawn from the same starting point. If a coin is the tossed (with two possible outcomes , head or tail), we draw the tree diagram as shown below.



Worked Example 1

QUESTION 4

Figures obtained from a city's police department seem to indicate that of all the motor vehicles reported stolen, 80% were stolen by syndicates to be sold off and 20% were stolen by individual persons for their own use.

Of those vehicles presumed stolen by syndicates:

- 24% were recovered within 48 hours
- 16% were recovered after 48 hours
- 60% were never recovered

Of those vehicles presumed stolen by individual persons:

- 38% were recovered within 48 hours
- 58% were recovered after 48 hours
- 4% were never recovered

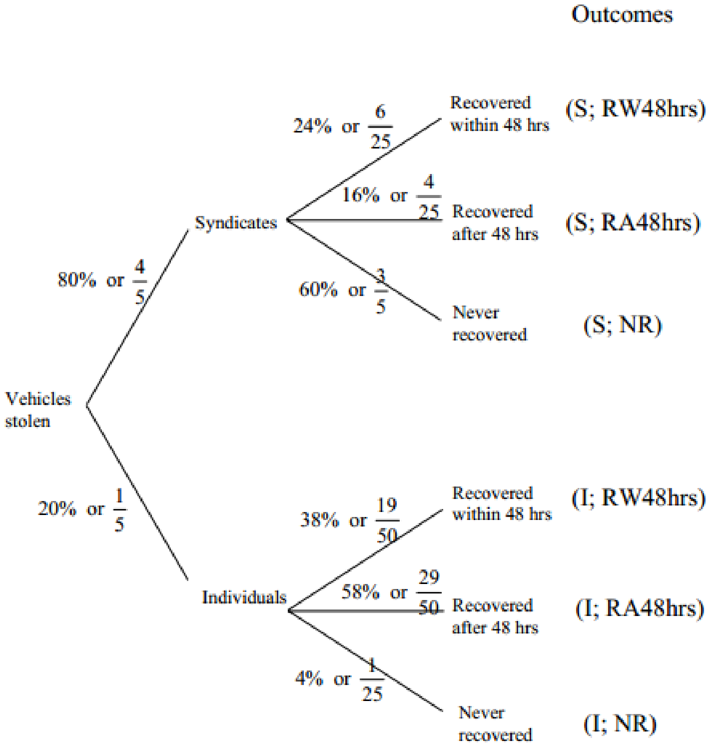
4.1 Draw a tree diagram for the above information.

4.2 Calculate the probability that if a vehicle were stolen in this city, it would be stolen by a syndicate and recovered within 48 hours.

4.3 Calculate the probability that a vehicle stolen in this city will not be recovered.

Solution

4.1



4.2 $P(S;\ RW48hrs\ ) = \frac{80}{100} \times \frac{24}{100} = \frac{1920}{10\ 000} = 0{,}192 = 19{,}2\%\ \ (0{,}19)$

OR

$P(S;\ RW48hrs\ ) = \frac{4}{5} \times \frac{6}{25} = \frac{24}{125}$

4.3 P(stolen and not recovered) =

$\left(\frac{80}{100} \times \frac{60}{100}\right) + \left(\frac{20}{100} \times \frac{4}{100}\right) = 0{,}488 = 48{,}8\% \quad (0{,}49)$

OR

P(stolen and not recovered) =

$\left(\frac{4}{5} \times \frac{3}{5}\right) + \left(\frac{1}{5} \times \frac{1}{25}\right) = \frac{12}{25} + \frac{1}{125} = \frac{61}{125}$

Worked Example 2 (Non-replacement)

4.2 There are 20 boys and 15 girls in a class. The teacher chooses individual learners at random to deliver a speech.

4.2.1 Calculate the probability that the first learner chosen is a boy.

4.2.2 Draw a tree diagram to represent the situation if the teacher chooses three learners, one after the other. Indicate on your diagram ALL possible outcomes.

4.2.3 Calculate the probability that a boy, then a girl and then another boy is chosen in that order.

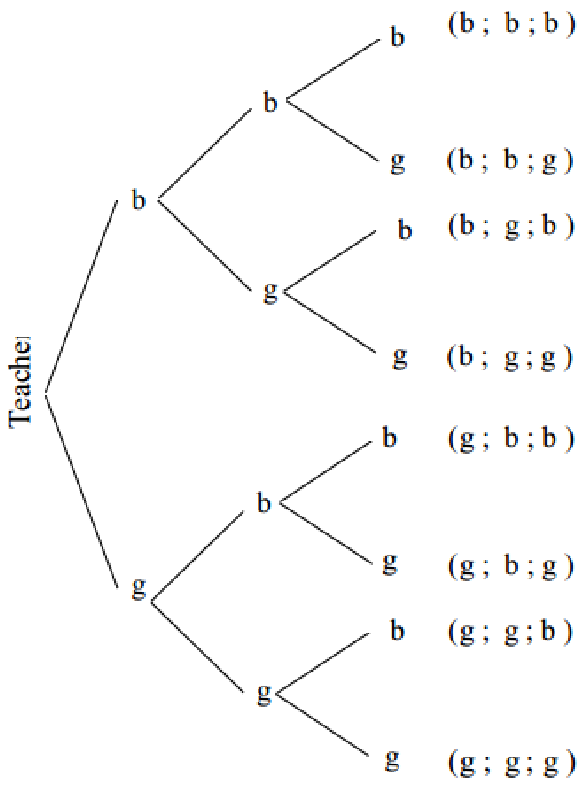
4.2.4 Calculate the probability that all three learners chosen are girls.

4.2.5 Calculate the probability that at least one of the learners chosen is a boy.

Solution

4.2.1 P(boy chosen first) = $\frac{20}{35} = \frac{4}{7} = 0{,}57$.

4.2.2 Outcomes



4.2.3 $P(b\,;g\,;b\,) = \frac{20}{35} \times \frac{15}{34} \times \frac{19}{33} = \frac{190}{1309} = 0{,}15$

4.2.4 $P(g\,;g\,;g\,) = \frac{15}{35} \times \frac{14}{34} \times \frac{13}{33} = \frac{13}{187} = 0{,}07$

4.2.5 P(at least one boy) $= 1 - P(\text{three girls chosen})$
$= 1 - 0{,}07$
$= 0{,}93$

SECTION 4: TWO-WAY CONTINGENCY TABLES

Two-way contingency tables are tools used to organize and display data involving **two categorical variables**. They help learners analyze and calculate probabilities based on real-world or survey data.

	A_1	B_1	**TOTAL**
B_1	p	q	$\boldsymbol{p+q}$
B_2	r	s	$\boldsymbol{r+s}$
TOTAL	$\boldsymbol{p+r}$	$\boldsymbol{q+s}$	$\boldsymbol{p+q+r+s=n}$

Below are the few examples on how to find probabilities using contingency table.

$$P(A_1) = \frac{\boldsymbol{p+r}}{n}$$

$$P(B_2) = \frac{\boldsymbol{r+s}}{n}$$

$$P(A_1 \cap B_2) = \frac{r}{n}$$

Worked Example 1

QUESTION 5

In a survey 1 530 skydivers were asked if they had broken a limb. The results of the survey were as follows:

	Broken a limb	**Not broken a limb**	**TOTAL**
Male	463	b	782
Female	a	c	d
TOTAL	913	617	1 530

5.1 Calculate the values of a, b, c and d.

5.2 Calculate the probability of choosing at random in the survey, a female skydiver who has not broken a limb.

5.3 Is being a female skydiver and having broken a limb independent? Use calculations, correct to TWO decimal places, to motivate your answer.

Solution

5.1	$a = 450$ $b = 319$ $c = 298$ $d = 748$
5.2	P(Female who has not broken a limb) $= \frac{298}{1530}$ $= \frac{149}{765}$
5.3	P(Female & broken a limb) $= \frac{450}{1530}$ $= \frac{5}{17}$ $= 0{,}2941176471...$ $= 0{,}29$ P(Female) × P(Broken a limb) $= \frac{748}{1530} \times \frac{913}{1530}$ $= 0{,}29$ The events of being female and having broken a limb are independent. If a candidate answers not independent due to the fact that the answers are not accurate to more than 2 decimal places, award full marks.

Worked Example 2

QUESTION 6

The data below was obtained from the financial aid office at a certain university.

	RECEIVING FINANCIAL AID	NOT RECEIVING FINANCIAL AID	TOTAL
Undergraduates	4 222	3 898	8 120
Postgraduates	1 879	731	2 610
TOTAL	6 101	4 629	10 730

6.1 Determine the probability that a student selected at random is ...

6.1.1 receiving financial aid.

6.1.2 a postgraduate student and not receiving financial aid.

6.1.3 an undergraduate student and receiving financial aid.

6.2 Are the events of being an undergraduate and receiving financial aid independent? Show ALL relevant workings to support your answer.

Solution

6.1.1	P(students receiving financial aid) $=\frac{6\ 101}{10730}$ $=0{,}57$	Answer only: Full marks
6.1.2	P(postgraduate not receiving financial aid) $=\frac{731}{10370}$ $=0{,}068$	Answer only: Full marks Also accept: $\frac{731}{2610}$
6.1.3	P(undergraduate receiving financial aid) $=\frac{4\,222}{10370}$ $=0{,}39$	Answer only: Full marks Also accept: $\frac{4\,222}{8\,120}$

6.2	Let UG be the event of being an undergraduate and RF be the event of receiving financial aid. P(UG and RF) $=\frac{4\ 222}{10\ 730}$ $=0{,}39$ P(UG) × P(RF) $=\frac{8\ 120}{10\ 730}\times\frac{6\ 101}{10\ 730}$ **OR** $=0{,}76\times0{,}57$ $=0{,}4332$ $=0{,}43$ P(UG and RF) ≠ P(UG) × P(RF) The event of being an undergraduate and receiving financial aid are NOT independent.

SECTION 5: ACTIVITIES OF PROBABILITY

Activity 1

QUESTION 11

A certain number of learners are sitting for examinations in Mathematics, Tourism and Geography.

- All these learners sit for at least one of these examinations.
- The total number of learners who sit for Mathematics (M), is 22.
- The total number of learners sitting for Tourism (T), is 16.
- The total number of learners sitting for Geography (G), is 18.
- 5 learners sit for Mathematics and Tourism, but not Geography.
- 4 learners sit for Mathematics and Geography, but not Tourism.
- 3 learners sit for Tourism and Geography, but not Mathematics.
- 6 learners sit for only Tourism.

11.1 Draw a Venn diagram to represent ALL the learners sitting for these examinations.

11.2 Calculate the probability that a learner, chosen at random, will sit for examinations in at least TWO of the subjects.

11.3 Determine if the events: sitting for examinations in Mathematics and sitting for examinations in Tourism are independent. Support your answer with the necessary calculations.

Activity 2

QUESTION 10

10.1 A and B are independent events. $P(A) = \frac{1}{3}$ and $P(B) = \frac{3}{4}$

Determine:

10.1.1 P(A and B) (2)

10.1.2 P(at least ONE event occurs) (2)

10.2 The probability that it will snow on the Drakensberg Mountains in June is 5%.

- When it snows on the mountains, the probability that the minimum temperature in Central South Africa will drop below 0 °C is 72%.
- If it does not snow on the mountains, the probability that the minimum temperature in Central South Africa will drop below 0 °C is 35%.

10.2.1 Represent the given information on a tree diagram. Clearly indicate the probabilities associated with EACH branch. (3)

10.2.2 Calculate the probability that the temperature in Central South Africa will NOT drop below 0 °C in June 2024. (3)

Activity 3

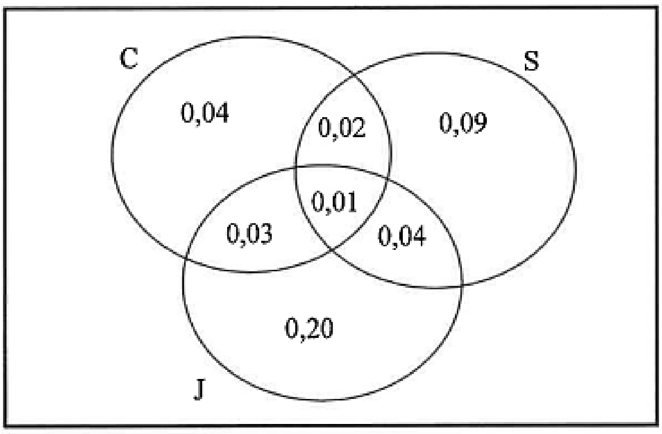
QUESTION 11

11.1 Two events, A and B, are such that:

- P(A) = 0,4
- P(A or B) = 0,52
- A and B are mutually exclusive

Calculate P(B). (2)

11.2 The items that a learner bought at a tuck shop were recorded over a period of time. The probabilities of the learner buying a sandwich (S), a chocolate (C) and a juice (J) are shown in the Venn diagram below.



11.2.1 What is the probability that the learner will buy a sandwich? (1)

11.2.2 Calculate the probability that the learner will buy at least two of the three items. (2)

11.2.3 Calculate the probability that the learner would NOT buy any of the three items. (2)

Activity 4

QUESTION 10

10.1 A group of people participated in a trial to test a new headache pill.

- 50% of the participants received the headache pill.
- 50% of the participants received a sugar pill.
- $\frac{2}{5}$ of the group receiving the headache pill were not cured.
- $\frac{3}{10}$ of the group receiving the sugar pill were cured.

10.1.1 Represent the given information on a tree diagram. Indicate on your diagram the probability associated with each branch as well as the outcomes. (3)

10.1.2 Determine the probability that a person chosen at random from the group will NOT be cured. (2)

10.2 Three events, A, B and C, are considered:
$P(A)=\frac{2}{5}$, $P(B)=\frac{1}{4}$ and $P(A \text{ or } B)=\frac{13}{20}$.

10.2.1 Are events A and B mutually exclusive? Support your answer with the necessary calculations. (2)

10.2.2 Determine P(only C), if it is further given that
$P(A \text{ or } C)=\frac{7}{10}$, $P(A \text{ and } C)=\frac{2}{5}$ and $2P(B \text{ and } C)=P(A \text{ and } C)$. (3)

10.2.3 Determine the probability that events A, B or C do NOT take place. (2)

Activity 5

QUESTION 10

10.1 A bag contains 7 yellow balls, 3 red balls and 2 blue balls. A ball is chosen at random from the bag and not replaced. A second ball is then chosen. Determine the probability that of the two balls chosen, one is red and the other is blue. (4)

10.2 Learners at a hostel may choose a meal and a drink for lunch. Their selections on a certain day were recorded and shown in the partially completed table below.

		MEAL		TOTAL
		SANDWICH (S)	**PASTA (P)**	
DRINK	Fruit Juice (F)	a	30	b
	Bottled Water (W)			
TOTAL		200		250

The probability of a learner choosing fruit juice and a sandwich on that day was 0,48.

10.2.1 Calculate the number of learners who chose fruit juice and a sandwich for lunch on that day. (1)

10.2.2 Is the choice of fruit juice independent of the choice of a sandwich for lunch on that day? Show ALL calculations to motivate your answer. (4)
[9]

Activity 6

QUESTION 2

One hundred and seventy-five movie critics were invited to preview a new movie. After seeing the movie, a survey was conducted and the results were recorded in a two-way contingency table.

	Age < 40	**Age ≥ 40**	**Totals**
Liked the movie	65	37	102
Did not like the movie	b	31	a
Totals	c	d	175

2.1 Calculate the values of a, b, c and d in the contingency table. (4)

2.2 A movie critic is selected at random. What is the probability that the critic was less than 40 years old and did not like the movie? (2)

2.3 Are the events, age of the critic and preference for the movie, independent? Support your answer with the appropriate calculations. (4)
[10]

Annexure A: Information Sheet

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$A = P(1+ni)$ $\quad A = P(1-ni)$ $\quad A = P(1-i)^n$ $\quad A = P(1+i)^n$

$T_n = a + (n-1)d$ $\quad S_n = \frac{n}{2}[2a + (n-1)d]$

$T_n = ar^{n-1}$ $\quad S_n = \frac{a(r^n - 1)}{r-1}$; $\quad r \neq 1$ $\quad S_\infty = \frac{a}{1-r}$; $-1 < r < 1$

$F = \frac{x[(1+i)^n - 1]}{i}$ $\quad P = \frac{x[1-(1+i)^{-n}]}{i}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $\quad \mathrm{M}\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$

$y = mx + c$ $\quad y - y_1 = m(x - x_1)$ $\quad m = \frac{y_2 - y_1}{x_2 - x_1}$ $\quad m = \tan\theta$

$$(x-a)^2 + (y-b)^2 = r^2$$

$In\ \Delta ABC\text{:}\ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ $\quad a^2 = b^2 + c^2 - 2bc.\cos A$ $\quad area\ \Delta ABC = \frac{1}{2}ab.\sin C$

$\sin(\alpha + \beta) = \sin\alpha.\cos\beta + \cos\alpha.\sin\beta$ $\quad \sin(\alpha - \beta) = \sin\alpha.\cos\beta - \cos\alpha.\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha.\cos\beta - \sin\alpha.\sin\beta$ $\quad \cos(\alpha - \beta) = \cos\alpha.\cos\beta + \sin\alpha.\sin\beta$

$\cos 2\alpha = \begin{cases} \cos^2\alpha - \sin^2\alpha \\ 1 - 2\sin^2\alpha \\ 2\cos^2\alpha - 1 \end{cases}$ $\quad \sin 2\alpha = 2\sin\alpha.\cos\alpha$

$\bar{x} = \frac{\sum x}{n}$ $\quad \sigma^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$

$P(A) = \frac{n(A)}{n(S)}$ $\quad P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$\hat{y} = a + bx$ $\quad b = \frac{\sum(x - \bar{x})(y - \bar{y})}{\sum(x - \bar{x})^2}$

Annexure B: Examination Guidelines

ELABORATION OF CONTENT/TOPICS

The purpose of the clarification of the topics is to give guidance to the teacher in terms of depth of content necessary for examination purposes. Integration of topics is encouraged as learners should understand Mathematics as a holistic discipline. Thus questions integrating various topics can be asked.

FUNCTIONS

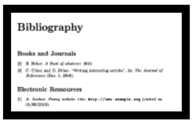
1. Candidates must be able to use and interpret functional notation. In the teaching process learners must be able to understand how $f(x)$ has been transformed to generate $f(-x)$, $-f(x)$, $f(x+a)$ $f(x)+a$, $af(x)$ and $x = f(y)$ where $a \in R$.
2. Trigonometric functions will ONLY be examined in PAPER 2.

DIFFERENTIAL CALCULUS

1. The following notations for differentiation can be used: $f'(x)$, D_x, $\frac{dy}{dx}$ or y'.
2. In respect of cubic functions, candidates are expected to be able to:
   - Determine the equation of a cubic function from a given graph.
   - Discuss the nature of stationary points including local maximum, local minimum and points of inflection.
   - Apply knowledge of transformations on a given function to obtain its image.
3. Candidates are expected to be able to draw and interpret the graph of the derivative of a function.
4. Surface area and volume will be examined in the context of optimisation.
5. Candidates must know the formulae for the surface area and volume of the right prisms. These formulae will NOT be provided on the formula sheet
6. If the optimisation question is based on the surface area and/or volume of the cone, sphere and/or pyramid, a list of the relevant formulae will be provided in that question. Candidates will be expected to select the correct formula from this list.

PROBABILITY

1. Dependent events are examinable but conditional probabilities are not part of the syllabus.
2. Dependent events in which an object is not replaced are examinable.



BIBLIOGRAPHY

1	**CAPS Document**
2	**Examination guidelines**
3	**Dbe November 2008 – 2024 Past Papers**
4	**Dbe June and Feb/Mar 2009 – 2024 Past Papers**
5	**Mind The Gap Textbook**
6	**Mind Action Series Textbook**
7	**JENN 2022 – 2024 Manuals**